COMMUTATIVE FORMAL GROUPS ARISING FROM SCHEMES

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Abstract. We prove the following criterion for the pro-representability of the deformation cohomology of a commutative formal Lie group. Let f be a flat and separated morphism between noetherian schemes. Assume that the target of f is flat over the integers. For a commutative formal Lie group E, we have the deformation cohomology of f with coefficients in E at our disposal. If the higher direct images of the tangent space of E are locally free and of finite rank then the deformation cohomology is pro-representable by a commutative formal Lie group.

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Introduction

Let $f : X \to S$ be a flat and separated morphism. The formal completion $\widehat{\mathrm{Pic}}_{X/S}$ of the relative Picard sheaf can be described as the first deformation cohomology $\Phi^1(X/S)$ with coefficients in the formal completion $\hat{\mathbb{G}}_m$ of the multiplicative

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group. Artin and Mazur, in their paper [AM77], study the higher degree analogues $\Phi^r(X/S)$, motivated by the idea that these sheaves exhibit a strong tendency to be pro-representable by formal Lie groups. Commutative formal Lie groups provide interesting invariants over a field of positive characteristic but poor invariants in characteristic zero where every commutative formal Lie group is isomorphic to a product of $\hat{\mathbb{G}}_m$.

The purpose of this paper is to give a convenient criterion for $\Phi^r(X/S)$ to be pro-representable when S is flat over $\mathrm{Spec}(\mathbb{Z})$. In order to state our theorem we need to introduce a bit of notation. We denote by $\Phi^r(X/S,E)$ the deformation cohomology of a commutative formal Lie group E over X (see 1.1.2) and $\Phi^r(X/S) = \Phi^r(X/S,\hat{\mathbb{G}}_m)$. The tangent space of E is denoted by TE, it is a locally free $\mathcal{O}_X$-module; for example, $T\hat{\mathbb{G}}_m = \mathcal{O}_X$.

Theorem 1 (Theorem 3.1.1)**.** *Let $f : X \to S$ be a flat and separated morphism between noetherian schemes. Suppose that S is flat over* $\mathrm{Spec}(\mathbb{Z})$*. Let E be a commutative formal Lie group over X. Suppose that R^rf_*TE is locally free and of finite rank for all $r \geq r_0$. Then $\Phi^r(X/S,E)$ is pro-representable by a commutative formal Lie group over S for all $r \geq r_0$.*

The tool which allows us to prove the theorem is Cartier theory. It turns out that the Cartier module associated to $\Phi^r(X/S)$ is $H^r(X,\mathbb{W}\mathcal{O}_X)$, where $\mathbb{W}\mathcal{O}_X$ is the sheaf of big Witt vectors, provided that S is affine. The pro-representability of $\Phi^r(X/S)$ can be therefore reduced to certain properties of $H^r(X,\mathbb{W}\mathcal{O}_X)$ which are made precise by Cartier theory.

In fact, one motivation for the paper is the existence of a big de Rham-Witt complex $\mathbb{W}\Omega^\bullet_X$ for arbitrary schemes due to Hesselholt and Madsen [HM01], [Hes]. Here $\mathbb{W}\mathcal{O}_X$ appears as a quotient of $\mathbb{W}\Omega^\bullet_X$, and in view of the Bloch-Illusie slope spectral sequence for smooth varieties over a perfect field one could expect that $H^*(X,\mathbb{W}\mathcal{O}_X)$ captures an interesting part of $H^*(X,\mathbb{W}\Omega^\bullet_X)$.

Our account of Cartier theory is Zink's book [Zin84] and we give an overview of the main results in Section 2.

In Section 4.1 we study the first interesting case where $S = \mathrm{Spec}(\mathbb{Z}[N^{-1}])$ and the formal Lie group $\Phi^r(X/\mathbb{Z}[N^{-1}])$ is one-dimensional. Our result (Proposition 4.1.11) states that the isomorphism class of $\Phi^r(X/\mathbb{Z}[N^{-1}])$ is determined by the slope < 1 part $H^r_{\mathrm{rig}}(X\otimes_{\mathbb{Z}}\mathbb{F}_p/\mathbb{Q}_p)_{<1}$ of the rigid cohomology of the reductions modulo p, where p runs over all primes not dividing N. For this, we use the correspondence of one-dimensional formal Lie groups and formal Dirichlet series due to Honda [Hon70]. For not necessarily one-dimensional groups the isomorphism class of $\Phi^r(X/\mathbb{Z}[N^{-1}])$ determines the slope < 1 part of the reductions, but we don't know the precise relation in dimension > 1.

In Section 4.2 we describe the relationship with the Gauss-Manin connection. If S is a smooth scheme over $\mathbb{Z}$, and $\mathfrak{X}$ is a commutative formal Lie group over S, then the Katz-Oda construction yields a connection

$$\nabla : H^i(\Omega^*_{\mathfrak{X}/S}) \to H^i(\Omega^*_{\mathfrak{X}/S}) \otimes_{\mathcal{O}_S} \Omega^1_{S/\mathbb{Z}} \qquad \text{for all } i.$$

Since in the invariant 1-forms (see A.1) are automatically closed, we may define $(H^1_{\mathrm{inv}}(\mathfrak{X}),\nabla)$ to be the smallest subobject of $(H^1(\Omega^*_{\mathfrak{X}/S}),\nabla)$ that contains the invariant 1-forms. As an application of a theorem of Stienstra [Sti91] we will prove the following statement.

Theorem 2 (Theorem 4.2.5)**.** *Let $f : X \to S$ be a smooth, projective morphism of relative dimension r. Suppose that S is smooth over* $\mathrm{Spec}(\mathbb{Z})$ *and suppose that $R^j f_* \Omega^i_{X/S}$ is locally free for all i, j. Then $(H^1_{\mathrm{inv}}(\Phi^i(X/S)), \nabla)$ is a subquotient of the Gauss-Manin connection $(H^{2r-i}_{\mathrm{dR}}(X/S), \nabla)$ for all $i \geq 0$.*

1. Artin-Mazur functor

1.1. **Deformation cohomology.**

1.1.1. Let X be a scheme. Let E be a sheaf of abelian groups on the big Zariski site of X. The *formal completion* $\hat{E}$ of E along its zero section is defined by

$$\hat{E}(Z) := \ker(E(Z) \to E(Z_{\mathrm{red}})).$$

The formal completion $\hat{E}$ is a sheaf on the big Zariski site again (cf. [AM77, II.1]).

Examples are $\hat{\mathbb{G}}_a$ and $\hat{\mathbb{G}}_m$; in both cases the sheaves are pro-represented by the formal scheme $\mathrm{Spf}(\mathcal{O}_X[|t|])$, the group law being $m^*(t) = t\hat{\otimes}1 + 1\hat{\otimes}t$ for $\hat{\mathbb{G}}_a$ and $m^*(t) = t\hat{\otimes}1 + 1\hat{\otimes}t - t\hat{\otimes}t$ for $\hat{\mathbb{G}}_m$.

1.1.2. Let $f : X \to S$ be a morphism of schemes. For every morphism $g : T \to S$ we define a sheaf $\tilde{E}_T$ on the small Zariski site of $X_T := X \times_S T$ as follows: let $\iota : X \times_S T_{\mathrm{red}} \to X \times_S T$ be the base change of $T_{\mathrm{red}} \to T$, we set

$$\tilde{E}_T := \ker(E \to \iota_* E),$$

i.e. $\tilde{E}_T(U) = \ker(E(U) \to E(U \times_T T_{\mathrm{red}}))$ for every open $U \subset X_T$. By denoting $f_T : X_T \to T$ the base change of f, we obtain for every integer $q \geq 0$ the sheaf $R^q f_{T*} \tilde{E}_T$ on T.

Definition 1.1.3. For a morphism $g : T \to S$ we set

$$\Phi^q(X/S, E)(g : T \to S) := \Gamma(T, R^q f_{T*} \tilde{E}_T).$$

The assignment $\Phi^q(X/S, E)$ defines a sheaf on the big Zariski site of S, which is called the *deformation cohomology* of X/S with coefficients in E [AM77, II.(1.4)]. Indeed, for $T' \xrightarrow{h} T \xrightarrow{g} S$, the natural map $(id_X \times h)^{-1} \tilde{E}_T \to \tilde{E}_{T'}$ induces

$$h^{-1} R^q f_{T*} \tilde{E}_T \to R^q f_{T'*} (id_X \times h)^{-1} \tilde{E}_T \to R^q f_{T'*} \tilde{E}_{T'},$$

equipping $\Phi^q(X/S, E)$ with the structure of a presheaf. Since

$$\Gamma(U, R^q f_{T*} \tilde{E}_T) = \Gamma(U, R^q f_{U*} \left(\tilde{E}_T \mid_{f_T^{-1} U} \right)) = \Gamma(U, R^q f_{U*} \tilde{E}_U),$$

for every open U of T, it is a sheaf.

For a morphism $h : Y \to X$ over S we get an induced morphism of sheaves

$$\Phi^q(h, E) : \Phi^q(X/S, E) \to \Phi^q(Y/S, E),$$

which is functorial in h.

Obviously, the deformation cohomology for E and its formal completion $\hat{E}$ agree:

$$\Phi^q(X/S, E) = \Phi^q(X/S, \hat{E}) \qquad \text{for all } q.$$

Remark 1.1.4. Artin and Mazur use the étale topology in [AM77, II] for the definition of the deformation cohomology. For the coefficients we will be considering étale and Zariski deformation cohomology agree (Lemma 1.1.9).

1.1.5. If E is formally smooth then, by definition, we have an exact sequence

$$0 \to \tilde{E}_T \to E_{|X_T} \to \iota_*(E_{|X_{T_{\mathrm{red}}}}) \to 0.$$

Therefore we obtain a long exact sequence

$$\begin{aligned} \ldots \to \Phi^q(X/S,E)_{|T} \to R^q f_{T*}(E_{|X_T}) \to R^q f_{T_{\mathrm{red}}*}(E_{|X_{T_{\mathrm{red}}}}) \to \\ \to \Phi^{q+1}(X/S,E)_{|T}, \end{aligned} \tag{1.1.1}$$

and $\Phi^*(X/S,E)$ controls the deformation of cohomology classes on T_{red} to classes on T. A classical case is $E = \mathbb{G}_m$ and $S = \mathrm{Spec}(R)$ with R a discrete valuation ring with uniformizer t and residue field k. Let $T_n = \mathrm{Spec}(R/t^n)$; we get an exact sequence:

$$\Phi^1(X/S,\hat{\mathbb{G}}_m)(R/t^n) \to \mathrm{Pic}(X_{T_n}) \to \mathrm{Pic}(X \otimes_R k) \to \Phi^2(X/S,\hat{\mathbb{G}}_m)(R/t^n).$$

For $L \in \mathrm{Pic}(X \otimes_R k)$ we obtain an obstruction class in $\varprojlim_n \Phi^2(X/S,\hat{\mathbb{G}}_m)(R/t^n)$ which becomes trivial if L lifts to $\varprojlim_n \mathrm{Pic}(X_{T_n})$.

1.1.6. Let $f : X \to S$ be a flat and proper morphism. Assume that f is cohomologically flat in dimension zero. Then Pic X/S is represented by an algebraic space whose formal completion along the zero section is $\Phi^1(X/S,\hat{\mathbb{G}}_m)$ (cf. [AM77, II.4]).

1.1.7. If E is a commutative formal group over X then we say that E is a *commutative formal Lie group* if every point $x \in X$ admits an open affine neighborhood $U = \mathrm{Spec}(A)$ and an isomorphism $E \times_X U \cong \mathrm{Spf}(A[|x_1,\ldots,x_d|])$ as formal schemes over $\mathrm{Spec}(A)$ such that the zero $0 \in (E \times_X U)(\mathrm{Spec}(A))$ identifies with the morphism $\mathrm{Spec}(A) \to \mathrm{Spf}(A[|x_1,\ldots,x_d|])$ given by $x_i \mapsto 0$ for all i.

For a commutative formal Lie group E we define the *tangent space* TE by

$$TE(U) := \ker(E(\mathrm{Spec}(\mathcal{O}_U[\epsilon]/\epsilon^2)) \to E(U))$$

for every open $U \subset X$. The tangent space is a locally free $\mathcal{O}_X$-module of finite rank. For example, $T\hat{\mathbb{G}}_m = \mathcal{O}_X = T\hat{\mathbb{G}}_a$.

1.1.8. Again, let $f : X \to S$ be a morphism and E a commutative formal Lie group over X. Let $g : T \to S$ be a morphism and suppose that $T = \mathrm{Spec}(R)$ for a noetherian ring R. Set $T_i = \mathrm{Spec}(R/\mathrm{nil}(R)^i)$ for all $i \geq 1$, and

$$\tilde{E}^i_T := \ker(E \to \iota_{i*}E),$$

with $\iota_i : X_{T_i} \to X_T$ the base change of $T_i \to T$. For an open $U \subset X_T$, we have $\tilde{E}^i_T(U) = \ker(E(U) \to E(U \times_T T_i))$. The sheaf $\tilde{E}_T$ admits the filtration

$$\tilde{E}_T = \tilde{E}^1_T \supset \tilde{E}^2_T \supset \cdots \supset \tilde{E}^n_T = 0,$$

provided that $\mathrm{nil}(R)^n = 0$.

Lemma 1.1.9. *For all $i \geq 1$, the quotient $\tilde{E}^i_T/\tilde{E}^{i+1}_T$ is a coherent $\mathcal{O}_{X_{T_{\mathrm{red}}}}$-module. More precisely,*

$$(\mathrm{nil}(R)^i \cdot \mathcal{O}_{X_T}/\mathrm{nil}(R)^{i+1} \cdot \mathcal{O}_{X_T}) \otimes_{\mathcal{O}_{X_T}} (id \times g)^*TE \xrightarrow{\cong} \tilde{E}^i_T/\tilde{E}^{i+1}_T.$$

Proof. We may assume that $S = T$. There is a natural injective morphism

$$\tilde{E}_S^i / \tilde{E}_S^{i+1} \to \ker(\iota_{i+1*}E \to \iota_{i*}E).$$

Locally one easily shows that the morphism is also surjective. We have a morphism

$$\begin{aligned}(\mathrm{nil}(R)^i \cdot \mathcal{O}_X / \mathrm{nil}(R)^{i+1} \cdot \mathcal{O}_X) \otimes_{\mathcal{O}_X} TE &\to \ker(\iota_{i+1*}E \to \iota_{i*}E) \\ n \otimes s &\mapsto \phi_n^*(s),\end{aligned}$$

where $\phi_n : X \times_S S_{i+1} \to \mathrm{Spec}(\mathcal{O}_X[\epsilon]/\epsilon^2)$ is defined over X by $\phi_n^*(\epsilon) = n$. Again, one checks locally that the morphism is an isomorphism. $\square$

1.2. **Representation by a formal group: reduction to nilpotent algebras.**

1.2.1. We will be mainly concerned with the following question when S is flat over $\mathrm{Spec}(\mathbb{Z})$.

Question 1.2.2. *Let E be a commutative formal Lie group over X. When is $\Phi^q(X/S, E)$ pro-representable by a commutative formal Lie group over S?*

For simplicity we will work with noetherian schemes.

A pro-representing formal scheme $\mathfrak{X}$ with an isomorphism $\mathfrak{X} \cong \Phi^q(X/S, E)$ is unique up to unique isomorphism. Indeed, if $J \subset \mathcal{O}_{\mathfrak{X}}$ is the maximal ideal of definition then the scheme $X_n := (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/J^n)$ represents the restriction of $\Phi^q(X/S, E)$ to the category of schemes T over S such that $\mathrm{nil}(\mathcal{O}_T)^n = 0$. Thus X_n is unique, which implies that $\mathfrak{X} = \varinjlim_n X_n$ is unique. In particular, Question 1.2.2 is local in S.

1.2.3. Let R be a commutative ring. For an R-algebra A we define a new R-algebra $N_R(A)$ by

$$N_R(A) := R \oplus \mathrm{nil}(A)/\{(r, -r); r \in \mathrm{nil}(R)\}.$$

The multiplication is given by $(r, n) \cdot (r', n') = (rr', rn' + r'n + nn')$. Obviously,

$$\mathrm{nil}(A) \xrightarrow{\cong} \mathrm{nil}(N_R(A)), \quad n \mapsto (0, n),$$

is an isomorphism. Clearly, N_R defines a functor

$$N_R : (R\text{-algebras}) \to (R\text{-algebras}).$$

Moreover, there is a morphism of functors $N_R \to id$ induced by the natural map

$$N_R(A) \to A, \quad (r, n) \mapsto r + n. \tag{1.2.1}$$

Let $\mathcal{C}$ be the full subcategory consisting of R-algebras A such that (1.2.1) is an isomorphism. Then N_R takes values in $\mathcal{C}$ and is right adjoined to the forgetful functor $\mathcal{C} \to (R\text{-algebras})$:

$$\mathrm{Hom}_R(B, A) = \mathrm{Hom}_R(B, N_R(A)) \quad \text{for all } B \in \mathcal{C} \text{ and any } R\text{-algebra } A.$$

Proposition 1.2.4. *Let $f : X \to S = \mathrm{Spec}(R)$ be a flat morphism between noetherian schemes. Let E be a commutative formal Lie group over X and let $g : \mathrm{Spec}(A) \to S$ be a morphism. We assume that A is noetherian. For all $q \geq 0$ there is a functorial isomorphism*

$$H^q(X_{\mathrm{Spec}(N_R(A))}, \tilde{E}_{\mathrm{Spec}(N_R(A))}) \xrightarrow{\cong} H^q(X_{\mathrm{Spec}(A)}, \tilde{E}_{\mathrm{Spec}(A)}). \tag{1.2.2}$$

Proof. By using (1.2.1) we obtain a functorial morphism (1.2.2). Set $T := \mathrm{Spec}(A)$; for every open $U \subset X$ we claim that

$$\Gamma(U_{\mathrm{Spec}(N_R(A))}, \tilde{E}_{\mathrm{Spec}(N_R(A))}) \to \Gamma(U_T, \tilde{E}_T) \tag{1.2.3}$$

is an isomorphism.

In fact, we may assume that $U = X$. Suppose first that X is separated. Then it is sufficient to prove (1.2.3) for every affine open $U = \mathrm{Spec}(B)$ in X. Moreover, we may assume that $E \times_X U \cong \mathrm{Spf}(B[|x_1, \dots, x_d|])$. Then (1.2.3) reads

$$\prod_{i=1}^{d} \ker(B \otimes_R N_R(A) \to B \otimes_R (N_R(A)/\mathrm{nil}(A))) \to \prod_{i=1}^{d} \ker(B \otimes_R A \to B \otimes_R (A/\mathrm{nil}(A))).$$

Since B is flat over R both sides equal $(B \otimes_R \mathrm{nil}(A))^d$.

If X is not separated then we still have (1.2.3) for every separated open U of X. Since an open of a separated scheme is separated again, we can simply use a covering of X by separated (e.g. affine) open sets to conclude (1.2.3) for X.

Let us prove that (1.2.2) is an isomorphism. Again, suppose first that X is separated. Since (1.2.3) holds it is sufficient to show that $H^q(X_T, \tilde{E}_T)$, for any noetherian affine scheme T, can be calculated by the cohomology of the Čech complex associated to an affine open covering. This follows because $\tilde{E}_T$ has a finite filtration such that the graded pieces are coherent (Lemma 1.1.9).

If X is not separated then we take a finite covering $\mathfrak{U}$ of X by affine open sets. By using the spectral sequence

$$E_1^{p,q} = \bigoplus_{i_0 < \cdots < i_p} H^q(\bigcap_{j=0}^{p} U_{i_j,T}, \tilde{E}_T) \Rightarrow H^{p+q}(X_T, \tilde{E}_T),$$

we reduce to the statement for the separated schemes $\bigcap_{j=0}^{p} U_{i_j}$. $\square$

1.2.5. Let R be a commutative algebra. By a *nilpotent R-algebra* $\mathcal{N}$ we mean an R-algebra $\mathcal{N}$ without 1-element, such that $\mathcal{N}^r = 0$ for an integer $r \geq 1$. There is an obvious R-algebra $R \oplus \mathcal{N}$ attached to every nilpotent R-algebra $\mathcal{N}$.

Let $f : X \to \mathrm{Spec}(R)$ be a morphism and E a commutative formal Lie group over X. For a nilpotent R-algebra $\mathcal{N}$ we define a sheaf $\tilde{E}_{\mathcal{N}}$ on the small Zariski site of $X_{\mathrm{Spec}(R \oplus \mathcal{N})}$ by

$$\begin{aligned} \tilde{E}_{\mathcal{N}} &= \ker(E_{|\mathrm{Spec}(R \oplus \mathcal{N})} \to \iota_* E_{|\mathrm{Spec}(R)}), \\ &= \ker(\tilde{E}_{\mathrm{Spec}(R \oplus \mathcal{N})} \to \iota_* \tilde{E}_{\mathrm{Spec}(R)}), \end{aligned}$$

where $\iota : \mathrm{Spec}(R) \to \mathrm{Spec}(R \oplus \mathcal{N})$ is induced by $R \oplus \mathcal{N} \to R, (r, n) \mapsto r$. We note that $X \to X_{\mathrm{Spec}(R \oplus \mathcal{N})}$ is topologically an isomorphism, and we can consider $\tilde{E}_{\mathcal{N}}$ as sheaf on X.

For an integer $q \geq 0$ we define

$$\begin{aligned} \Psi^q(X/S, E) &: (\text{nilpotent } R\text{-algebras}) \to (\text{abelian groups}) \\ \Psi^q(X/S, E)(\mathcal{N}) &= H^q(X, \tilde{E}_{\mathcal{N}}). \end{aligned} \tag{1.2.4}$$

Notation 1.2.6. We say that a functor

$$\Psi : (\text{nilpotent } R\text{-algebras}) \to (\text{abelian groups})$$

is pro-represented by the formal group $\mathfrak{X}$ over R if

$$\Psi \cong [\mathcal{N} \mapsto \ker(\mathfrak{X}(R \oplus \mathcal{N}) \to \mathfrak{X}(R))].$$

Proposition 1.2.7. *Let $q \geq 0$ be an integer. Let $f : X \to S = \mathrm{Spec}(R)$ be a flat morphism between noetherian schemes. Let E be a commutative formal Lie group over X. If $\Psi^q(X/S, E)$ is pro-represented by a commutative formal Lie group $\mathfrak{X}$ (over S) then the restriction of $\Phi^q(X/S, E)$ to the category of noetherian schemes (over S) is pro-represented by $\mathfrak{X}$.*

Proof. For a noetherian R-algebra A we have functorial isomorphisms

$$\Psi^q(X/S, E)(\mathrm{nil}(A)) \xrightarrow{\cong} H^q(X_{\mathrm{Spec}(N_R(A))}, \tilde{E}) \xrightarrow{\cong} H^q(X_{\mathrm{Spec}(A)}, \tilde{E}).$$

The first isomorphism is obvious and the second follows from Proposition 1.2.4. On the other hand, $\ker(\mathfrak{X}(R \oplus \mathrm{nil}(A)) \to \mathfrak{X}(R)) \cong \mathfrak{X}(A)$. Therefore the functor

$$(R\text{-alg}) \to (\text{abelian groups}), \quad A \mapsto H^q(X_{\mathrm{Spec}(A)}, \tilde{E}_{\mathrm{Spec}(A)}),$$

is pro-represented by $\mathfrak{X}$.

We have a morphism of functors

$$[A \mapsto H^q(X_{\mathrm{Spec}(A)}, \tilde{E})] \to [A \mapsto \Phi^q(X/S, E)(\mathrm{Spec}(A))], \tag{1.2.5}$$

and the right hand side is the sheafication of the left hand side. Since $\mathfrak{X}$ defines a sheaf, the restriction of $\Phi^q(X/S, E)$ to the category of noetherian affine schemes is represented by $\mathfrak{X}$. Again, since $\Phi^q(X/S, E)$ and $\mathfrak{X}$ are both sheaves we obtain $\Phi^q(X/S, E) \cong \mathfrak{X}$. □

2. Cartier theory

For the proof of the main theorem we will need Cartier theory as developed by Zink [Zin84]. In this section we give an overview of the results in [Zin84, III].

2.1. **Cartier modules.**

2.1.1. Let R be a commutative ring. For a functor

$$F : (\text{nilpotent } R\text{-algebras}) \to (\text{abelian groups}), \tag{2.1.1}$$

we set

$$F(xR[|x|]) := \varprojlim_n F(xR[x]/x^n R[x]).$$

The assignment $C : F \mapsto F(xR[|x|])$ is functorial. For a morphism $\Phi : F \to G$ of functors we denote by $\Phi_{R[|x|]} : F(xR[|x|]) \to G(xR[|x|])$ the induced morphism. The first theorem in Cartier theory states that C is representable provided that we restrict to left exact functors. A functor F as in (2.1.1) is called left exact if for every short exact sequence

$$0 \to \mathcal{N}_1 \xrightarrow{\phi} \mathcal{N}_2 \xrightarrow{\psi} \mathcal{N}_3 \to 0 \tag{2.1.2}$$

of nilpotent R-algebras (i.e. ϕ and ψ are morphism of R-algebras and (2.1.2) is exact in the sense of R-modules) the induced sequence

$$0 \to F(\mathcal{N}_1) \xrightarrow{F(\phi)} F(\mathcal{N}_2) \xrightarrow{F(\psi)} F(\mathcal{N}_3)$$

is exact. Right exact and exact functors are defined in the same way.

Define the functor Λ by

$$\Lambda : (\text{nilpotent } R\text{-algebras}) \to (\text{abelian groups}),$$
$$\Lambda(\mathcal{N}) := \{1 + n_1 t + n_2 t^2 + \cdots + n_r t^r \mid r \geq 0, n_i \in \mathcal{N}\} \subset (R \oplus \mathcal{N})[t]^\times.$$

Obviously, Λ is exact.

Theorem 2.1.2. [Zin84, 3.5] *Let H be a left exact functor. The morphism of abelian groups*

$$\mathrm{Hom}(\Lambda, H) \xrightarrow{\cong} H(xR[|x|]), \quad \Phi \mapsto \Phi_{R[|x|]}(1-xt),$$

is an isomorphism, where $1-xt \in \Lambda(xR[|x|])$.

In the theorem, $\mathrm{Hom}(\Lambda, H)$ denotes the natural transformations from Λ to H.

Definition 2.1.3 (Cartier ring)**.** Let R be a commutative ring. We set

$$\mathbb{E}_R := \mathrm{End}(\Lambda)^{op},$$

where $(.)^{op}$ denotes the opposite ring. We call $\mathbb{E}_R$ the *Cartier ring.*

Theorem 2.1.2 implies immediately that $H(xR[|x|])$ comes equipped with a natural left $\mathbb{E}_R$-module structure. We call $H(xR[|x|])$ the *Cartier module* attached to H.

2.1.4. Let $\mathfrak{X}$ be a commutative formal Lie group over R. We get an exact functor

$$\begin{aligned}(\text{nilpotent } R\text{-algebras}) &\to (\text{abelian groups})\\ \mathcal{N} &\mapsto \ker(\mathfrak{X}(R\oplus\mathcal{N}) \to \mathfrak{X}(R))\end{aligned}$$

which will be denoted by $\mathfrak{X}$ again. Therefore it makes sense to speak of the Cartier module of a commutative formal Lie group.

We note that the assignment

$$\mathfrak{X} \mapsto [\mathcal{N} \mapsto \ker(\mathfrak{X}(R\oplus\mathcal{N}) \to \mathfrak{X}(R))]$$

is a fully faithful functor from the category of commutative formal Lie groups over R to the category of functors from nilpotent R-algebras to abelian groups.

2.2. Cartier ring and V-reduced modules.

2.2.1. Let R be a commutative ring. Recall that

$$\mathbb{E}_R^{op} = \mathrm{End}(\Lambda) \xrightarrow{\cong,\lambda} \Lambda(xR[|x|]),$$

(Definition 2.1.3, Theorem 2.1.2). We obtain the following elements in $\mathbb{E}_R$:

$$V_n := \lambda^{-1}(1-x^n t), \quad F_n := \lambda^{-1}(1-xt^n), \qquad \text{for } n \geq 1, \tag{2.2.1}$$

$$[c] := \lambda^{-1}(1-cxt), \qquad \text{for } c \in R. \tag{2.2.2}$$

Theorem 2.2.2. [Zin84, 3.12] *Every element $\xi \in \mathbb{E}_R$ has a unique representation*

$$\xi = \sum_{n,m\geq 1} V_n[a_{n,m}]F_m,$$

with $a_{n,m} \in R$, and for every fixed n almost all $a_{n,m}$ vanish; in other words, $a_{n,m} = 0$ for $m \geq m_0(n)$ with $m_0(n)$ depending on n.

Proposition 2.2.3. [Zin84, 3.13] *The following relations hold in $\mathbb{E}_R$:*

$$\begin{aligned}
&F_1 = 1 = V_1 && F_nV_n = n && \textit{for } n \geq 1,\\
&[c]V_n = V_n[c^n] && F_n[c] = [c^n]F_n && \textit{for } n \geq 1,\ c \in R,\\
&V_mV_n = V_{n\cdot m} && F_nF_m = F_{n\cdot m} && \textit{for } m,n \geq 1,\\
& && F_nV_m = V_mF_n && \textit{if } \gcd(m,n) = 1.
\end{aligned}$$

For $c_1, c_2 \in R$ we have

$$[c_1][c_2] = [c_1c_2]$$
$$[c_1 + c_2] = [c_1] + [c_2] + \sum_{n=2}^{\infty} V_n[a_n(c_1, c_2)]F_n,$$

with $a_n \in \mathbb{Z}[X_1, X_2]$.

The subset $\{\sum_{n\geq 1} V_n[c_n]F_n \mid \forall n : c_n \in R\}$ of $\mathbb{E}_R$ defines a subring $\mathbb{W}(R)$ and is called the ring of *big Witt vectors* of R.

Example 2.2.4. We have an isomorphism of $\mathbb{E}_R$-modules

$$\mathbb{E}_R/\mathbb{E}_R \cdot (F_n - 1 \mid n \in \mathbb{Z}_{\geq 2}) \to \hat{\mathbb{G}}_m(R[|x|])$$
$$1 \mapsto 1 - x, \quad V_n \mapsto 1 - x^n, \quad [c] \mapsto 1 - cx.$$

In other words, the Cartier module of $\hat{\mathbb{G}}_{m,R}$ is $\mathbb{E}_R$ modulo the (left) ideal generated by the elements $F_n - 1$ where n runs through all natural numbers. Via

$$\mathbb{W}(R) \to \mathbb{E}_R/\mathbb{E}_R \cdot (F_n - 1 \mid n \in \mathbb{Z}_{\geq 2}) \to \hat{\mathbb{G}}_m(R[|x|])$$

we obtain an isomorphism $\mathbb{W}(R) \cong \hat{\mathbb{G}}_m(R[|x|])$ of $\mathbb{W}(R)$-modules.

For the Cartier module of $\hat{\mathbb{G}}_a$ we have an isomorphism

$$\mathbb{E}_R/\mathbb{E}_R \cdot (F_n \mid n \in \mathbb{Z}_{\geq 2}) \to \hat{\mathbb{G}}_a(R[|x|])$$
$$1 \mapsto x, \quad V_n \mapsto x^n, \quad [c] \mapsto cx.$$

More explicitly, with the identification $\hat{\mathbb{G}}_a(R[|x|]) = xR[|x|]$, we get
(2.2.3)

$$V_n(g(x)) = g(x^n), \quad F_n(cx^i) = \begin{cases} ncx^{\frac{i}{n}} & \text{if } n \text{ divides } i, \\ 0 & \text{otherwise,} \end{cases} \quad [c](g(x)) = g(cx).$$

Definition 2.2.5. [Zin84, 3.9] A *V-reduced* module is a left $\mathbb{E}_R$-module M together with a descending filtration

$$\cdots \subset M^n \subset \cdots \subset M^1 = M$$

by abelian subgroups satisfying the following properties:

(1) For all positive integers n, m and $c \in R$: $V_m[c]M^n \subset M^{nm}$.
(2) For all positive integers m, n there exists r such that $F_m M^r \subset M^n$.
(3) For all $m \geq 1$, the map $V_m : M/M^2 \to M^m/M^{m+1}$ is a bijection.
(4) The map $M \xrightarrow{\cong} \varprojlim_n M/M^n$ is a bijection.

A morphism of V-reduced modules is a morphism of $\mathbb{E}_R$-modules which respects the filtrations.

In this way we obtain the category of V-reduced modules which we denote by $\mathcal{C}V_{red}$. Clearly, $\mathcal{C}V_{red}$ is an additive category. The interest in V-reduced modules comes from the fact that the Cartier module M_H attached to an exact functor H can be viewed as V-reduced module; the filtration being

$$M_H^n := \mathrm{im}(H(x^n R[|x|]) \to H(xR[|x|])),$$

with $H(x^n R[|x|]) = \varprojlim_m H(x^n R[|x|]/x^m R[|x|])$. In particular, $\mathbb{E}_R$ is a V-reduced module. In order to avoid confusion we will write $\mathbb{E}_{R,n}$ for the filtration $\mathbb{E}_R^n$.

2.2.6. Let M be a V-reduced module. Let $\{x_\alpha\}_{\alpha \in M/M^2}$ be a system of representatives for M/M^2 in M, i.e. $x_\alpha = \alpha$ modulo M^2. Then every $x \in M$ has a unique representation

$$x = \sum_{n=1}^{\infty} V_n x_{\alpha_n}.$$

Moreover, we have

$$M^n = \sum_{i=n}^{\infty} V_i M.$$

2.2.7. The quotient M/M^2 comes equipped with an R-module structure defined by $c \cdot m := [c] \cdot m$. We obtain an additive functor

$$T : \mathcal{C}V_{red} \to (R\text{-modules}), \quad TM = M/M^2.$$

2.3. **Main theorem of Cartier theory.**

2.3.1. For a right $\mathbb{E}_R$-module N, we define

$$N_s := \{x \in N; x \cdot \mathbb{E}_{R,s} = 0\}$$

for all integers $s \geq 1$ (recall that $\mathbb{E}_{R,s} = \sum_{i=s}^{\infty} V_i \cdot \mathbb{E}_R$). If

$$N = \bigcup_{s \geq 1} N_s$$

then N is called a *torsion module*.

For a V-reduced module M the abelian subgroups

$$\text{image}(N_s \otimes_{\mathbb{Z}} M^s) \subset N \otimes_{\mathbb{E}_R} M, \quad \text{for } s \geq 1,$$

define an increasing filtration. Indeed, $n_s \otimes m_s = n_s \otimes (V_s x + m_{s+1}) = n_s \otimes m_{s+1}$, with $m_{s+1} \in M^{s+1}$. We set

$$(N \otimes_{\mathbb{E}_R} M)_\infty := \bigcup_{s \geq 1} \text{image}(N_s \otimes_{\mathbb{Z}} M^s)$$

Definition 2.3.2. [Zin84, 3.20] For a right $\mathbb{E}_R$-module N and a V-reduced module M we call the abelian group

$$N \bar{\otimes} M := N \otimes_{\mathbb{E}_R} M / (N \otimes_{\mathbb{E}_R} M)_\infty$$

the *reduced tensor product* of N with M.

The reduced tensor product is functorial in both arguments. If $N_1 \to N_2 \to N_3 \to 0$ is an exact sequence of torsion right $\mathbb{E}_R$-modules then

$$N_1 \bar{\otimes} M \to N_2 \bar{\otimes} M \to N_3 \bar{\otimes} M \to 0$$

is exact [Zin84, 3.20].

Proposition 2.3.3. [Zin84, 3.22] *Let $\mathcal{N}$ be a nilpotent R-algebra. Then $\Lambda(\mathcal{N})$ is a torsion module.*

2.3.4. For every V-reduced module M we define a functor $\Lambda\bar{\otimes}M$ by

$$\Lambda\bar{\otimes}M : (\text{nilpotent } R\text{-algebras}) \to (\text{abelian groups}), \qquad \mathcal{N} \mapsto \Lambda(\mathcal{N})\bar{\otimes}M.$$

Proposition 2.3.5. [Zin84, 3.26] *Let M be a V-reduced module. Suppose that M/M^2 is a flat R-module. Then $\Lambda\bar{\otimes}M$ is an exact functor.*

Definition 2.3.6. [Zin84, 3.27] A V-reduced module M such that M/M^2 is flat, will be called a *V-flat* module. We denote by $\mathcal{C}V_{fl}$ the full subcategory of $\mathcal{C}V_{red}$ consisting of V-flat modules.

2.3.7. A functor

$$F : (\text{nilpotent } R\text{-algebras}) \to (\text{abelian groups}),$$

is said to commute with arbitrary direct sums if the following property holds. For every integer $n \geq 1$ and for every sequence $(\mathcal{N}_i)_{i\in I}$ of nilpotent R-algebra with $\mathcal{N}_i^n = 0$ for all $i \in I$, indexed by a set I, the natural map

$$\bigoplus_{i\in I} F(\mathcal{N}_i) \to F(\bigoplus_{i\in I} \mathcal{N}_i)$$

is required to be an isomorphism.

Theorem 2.3.8. [Zin84, 3.28] *The functor $M \mapsto \Lambda\bar{\otimes}M$ induces an equivalence of categories between V-flat modules on the one hand, and exact functors from nilpotent R-algebras to abelian groups, which commute with arbitrary direct sums, on the other hand. The functor*

$$F \mapsto (M_F = F(xR[|x|]), M_F^n = \operatorname{im}(F(x^nR[|x|]) \to F(xR[|x|]))),$$

defines a quasi inverse.

2.3.9. We will need Theorem 2.3.8 in order to show that certain functors are exact and commute with direct sums. The relation with representability is given by the following corollary.

Proposition 2.3.10. [Zin84, 2.32] *Let H : (nilpotent R-algebras) $\to$ (ab. groups) be an exact functor which commutes with arbitrary direct sums. Suppose that $H(xR[x]/x^2R[x])$ is a free and finitely generated R-module. Then H is represented by a commutative formal Lie group $\mathfrak{X}$ in the sense of Notation 1.2.6.*

2.3.11. Let B be a flat R-algebra. Let E be a commutative formal Lie group over $\operatorname{Spec}(B)$ such that there exists an isomorphism $E \xrightarrow{\tau,\cong} \operatorname{Spf}(B[|x_1, \ldots, x_d|])$ as formal schemes. For us the most important example of an exact functor which commutes with direct sums is given by

$$F_{B,E} : (\text{nilpotent } R\text{-algebras}) \to (\text{abelian groups}), \qquad F_{B,E}(\mathcal{N}) = \ker(E(B \otimes_R (R \oplus \mathcal{N})) \to E(B))$$

Since B is flat over R the exactness follows from

$$F(\mathcal{N}) \xrightarrow{\tau,\cong \text{ as sets}} (B \otimes_R \mathcal{N})^d. \tag{2.3.1}$$

Since we already know that $F_{B,E}$ is exact and thus commutes with finite direct sums it is sufficient to show that

$$\bigcup_{\substack{J\subset I\\ J \text{ finite}}} F_{B,E}(\oplus_{i\in J}\mathcal{N}_i) = F_{B,E}(\oplus_{i\in I}\mathcal{N}_i)$$

in order to prove the compatibility with arbitrary direct sum. Again, this follows easily from (2.3.1).

The Cartier module attached to $F_{B,\hat{\mathbb{G}}_m}$ is $\mathbb{W}(B)$, the *big Witt vectors* of B.

2.3.12. Let $\mathfrak{X}$ be a commutative formal Lie group over R. Suppose that there exists an isomorphism $\mathfrak{X} \xrightarrow{\tau} \mathrm{Spf}(R[|x_1, \ldots, x_d|])$ as formal schemes over R. We denote by $e_i \in \mathfrak{X}(R[|x|])$ the curve defined by $\tau^{-1} \circ \xi_i$ and $\xi_i^*(x_j) = \delta_{i,j}x$.

Let M be the Cartier module of $\mathfrak{X}$, it is a V-flat module. Since

$$\{\sum_{j=1}^{d}[c_j] \cdot e_j \mid c_j \in R\}$$

is a system of representatives for $M/M^2 = T\mathfrak{X}$, every element $m \in M$ has a unique representation

$$m = \sum_{j=1}^{d}\sum_{i=1}^{\infty} V_i[c_{i,j}]e_j$$

(see 2.2.6). For an integer $n \geq 2$ we thus obtain

$$F_n e_k = \sum_{j=1}^{d} g_{n,j}e_j$$

for unique elements $g_{n,j} = \sum_{i=1}^{\infty} V_i[c_{n,i,j}]$.

In $\oplus_{j=1}^{d}\mathbb{E}_R\tilde{e}_j$ let I be the left $\mathbb{E}_R$-submodule generated by $F_n\tilde{e}_k = \sum_{j=1}^{d} g_{n,j}\tilde{e}_j$ for $n \geq 2, k = 1, \ldots, d$. Then

$$\oplus_{j=1}^{d}\mathbb{E}_R\tilde{e}_j/I \to M, \quad \tilde{e}_j \mapsto e_j$$

is an isomorphism of $\mathbb{E}_R$-modules.

2.3.13. *Local Cartier theory.* Let R be a ring and let p be a prime. Assume that every prime ℓ different from p is invertible in R; then ℓ is also invertible in $\mathbb{E}_R$ [Zin84, p.66]. The element $\epsilon \in \mathbb{E}_R$ defined by

$$\epsilon := \prod_{\ell \neq p}(1 - \frac{1}{\ell}V_\ell F_\ell)$$

is idempotent [Zin84, 4.11]. We set

$$\mathbb{E}_{R,p} := \epsilon \cdot \mathbb{E}_R \cdot \epsilon,$$

and call $\mathbb{E}_{R,p}$ the *p-typical Cartier ring.* We define the following elements in $\mathbb{E}_{R,p}$:

$$\begin{aligned} V &:= \epsilon V_p \epsilon = \epsilon V_p = V_p \epsilon, \\ F &:= \epsilon F_p \epsilon = \epsilon F_p = F_p \epsilon, \\ [x]_p &:= \epsilon[x]\epsilon = \epsilon[x] = [x]\epsilon, \qquad \text{for all } x \in R. \end{aligned}$$

Every element $\xi \in \mathbb{E}_{R,p}$ has a unique representation

$$\xi = \sum_{r,s \geq 0} V^r[x_{r,s}]F^s,$$

with $x_{r,s} \in R$, and for fixed r almost all $x_{r,s}$ vanish [Zin84, 4.17]. We have

$$\begin{aligned} &[1]_p = 1 && FV = p \\ &[x]_p V = V[x^p]_p && F[x]_p = [x^p]_p F. \end{aligned}$$

An $\mathbb{E}_{R,p}$-module M is called V-reduced if the following conditions hold:

(a) The map $V : M \to M$ is injective.
(b) The map $M \to \varprojlim_n M/V^n M$ is an isomorphism.

Theorem 2.3.14. [Zin84, 4.22]. *The functor $M \mapsto \epsilon M$ is an equivalence of categories*

$$\mathcal{C}V_{red} \to (V\text{-reduced } \mathbb{E}_{R,p}\text{-modules}).$$

For the tangent spaces we have $TM = M/M^2 = \epsilon M/V\epsilon M$.

3. Representation by formal groups

3.1. **Formulation of the main theorem.** Let $f : X \to S$ be a morphism of schemes. Let E be a commutative formal Lie group over X (see 1.1.7), e.g. $E = \hat{\mathbb{G}}_m$ the formal completion of $\mathbb{G}_m$ (see 1.1.1). The tangent space of E is denoted by TE. Recall Definition 1.1.3 for the Artin-Mazur functor $\Phi^q(X/S, E)$.

Theorem 3.1.1. *Let $f : X \to S$ be a flat separated morphism between noetherian schemes. Suppose that S is flat over* $\mathrm{Spec}(\mathbb{Z})$*. Let E be a commutative formal Lie group over X. Suppose that $R^q f_* TE$ is locally free and of finite rank for all $q \geq q_0$. Then the restriction of $\Phi^q(X/S, E)$ to the category of noetherian schemes (over S) is pro-representable by a commutative formal Lie group with tangent space $R^q f_* TE$ for all $q \geq q_0$.*

In the case $S = \mathrm{Spec}(R)$, we will show that the Cartier module of $\Phi^q(X/S, \hat{\mathbb{G}}_m)$ is given by $H^q(X, \mathbb{W}\mathcal{O}_X)$ where $\mathbb{W}\mathcal{O}_X$ is the sheaf of big Witt vectors.

3.2. **V-reduced modules revisited.** For the proof of Theorem 3.1.1 we will need some properties of V-reduced modules (see Definition 2.2.5) which we collect in this section.

3.2.1. Recall from Section 2.2 that we have a faithful functor

$$\imath : \mathcal{C}V_{red} \to (\text{left } \mathbb{E}_R\text{-modules})$$

by forgetting the filtration attached to a V-reduced module. Of course, left $\mathbb{E}_R$-modules form an abelian category, but $\mathcal{C}V_{red}$ is only an additive category. We say that a sequence of V-reduced modules is exact if it is an exact sequence after applying $\imath$.

Lemma 3.2.2. [Zin84, 3.24] *Let $g : M \to M''$ be a morphism in $\mathcal{C}V_{red}$ such that $\imath(g)$ is an epimorphism. Then $M' := \ker(g)$ exists and $\imath(M') = \ker(\imath(g))$. Moreover, the following sequence is exact:*

$$0 \to TM' \to TM \to TM'' \to 0.$$

Lemma 3.2.3. *Let $f : M' \to M$ be a morphism in $\mathcal{C}V_{red}$. Set $N'' := \mathrm{coker}(\imath(f))$ and $N''^2 := \mathrm{image}(M^2 \to N'')$. Suppose that N''/N''^2 is torsion free, i.e. for all $s \in \mathbb{Z}\backslash\{0\}$ and $n \in N''/N''^2$: $sn = 0$ implies $n = 0$. Then $M'' := \mathrm{coker}(f)$ exists and $\imath(M'') = N''$. Moreover, the following sequence is exact:*

$$TM' \to TM \to TM'' \to 0.$$

Proof. We define $N''^n := \text{image}(M^n \to N'')$ and equip $M'' := N''$ with the filtration $M''^n := N''^n$. We need to show that M'' satisfies the conditions 2.2.5(1–4), but (1) and (2) are obvious.

For (3). Surjectivity follows immediately from the definition. For the injectivity let $m'' \in M''$ such that $V_n m'' \in M''^{n+1}$, and choose a lifting $m \in M$ of m''. Since $V_n m \in M^{n+1} + f(M')$ and $F_n M^{n+1} \subset M^2$ we get

$$F_n V_n m = n \cdot m \in M^2 + f(M').$$

Thus $n \cdot m'' \in M''^2$ which implies $m'' \in M''^2$, because M''/M''^2 is torsion free.

For (4). Choose a set theoretic section $s : M''/M''^2 \to M$ with $s(0) = 0$. Denoting by $g : M \to M''$ the morphism to the cokernel (as $\mathbb{E}_R$-modules) we have $g(s(m'')) = m'' + M''^2$ for all $m'' \in M''$. We claim that every $m'' \in M''/M''^n$ has a representation

$$m'' = g(\sum_{i=0}^{n-1} V_i s(m''_i)) \mod M''^n \tag{3.2.1}$$

with $m''_i \in M''/M''^2$ being unique. The uniqueness and the existence follow easily from property (3) by induction.

In this way an element m'' in $\varprojlim_n M''/M''^n$ gives rise to a series $\sum_{i\geq 0} V_i s(m''_i)$ which defines a lifting of m'' in M. Therefore $M'' \to \varprojlim_n M''/M''^n$ is surjective.

Denote by $\pi : M \to M/M^2$ the projection. Choose a map of sets

$$s' : \text{image}(M'/M'^2 \xrightarrow{f} M/M^2) \to M'$$

such that for all $m \in \text{image}(M'/M'^2 \to M/M^2)$ the equality $\pi(f(s'(m))) = m$ holds. Define

$$t : M/M^2 \to M, \quad t(m) = (s \circ g)(m) + f(s'(m - (\pi \circ s \circ g)(m))).$$

Clearly, t is a section of π. Thus every $m \in M$ has a unique representation

$$m = \sum_{i\geq 0} V_i t(m_i), \tag{3.2.2}$$

with $m_i \in M/M^2$. Suppose $m'' \in M''$ is contained in the kernel of $M'' \to \varprojlim_n M''/M''^n$. Choose a lift $m \in M$ of m'' and write m as in (3.2.2). The uniqueness in (3.2.1) implies $g(m_i) = 0$ for all i. Since the series

$$\sum_{i\geq 0} V_i s'(m_i - (\pi \circ s \circ g)(m_i))$$

defines an element in M' we conclude that $m \in f(M')$ and $m'' = 0$. □

Lemma 3.2.4. *Let C be a bounded from above complex of V-flat modules. Let i_0 be an integer. Assume the following conditions.*

(1) *For all $i \geq i_0$, $H^i(TC)$ is an R-flat module.*
(2) *R is flat over $\mathbb{Z}$.*

Then the following holds.

(i) *For all $i \geq i_0$, $Z^iC := \ker(C^i \to C^{i+1})$ exists in $\mathcal{C}V_{red}$ and is a V-flat module. Moreover, $\iota(Z^iC) = Z^i(\iota(C))$ and $T(Z^iC) = Z^i(TC)$.*
(ii) *For all $i \geq i_0$, $H^iC := \text{coker}(C^{i-1} \to Z^iC)$ exists and is a V-flat module. Moreover, $\iota(H^iC) = H^i(\iota(C))$ and $T(H^iC) = H^i(TC)$.*

(iii) *For all* $i \geq i_0$, $B^iC := \ker(Z^iC \to H^iC)$ *exists and is a* V*-flat module. Moreover,* $\iota(B^iC) = B^i(\iota(C))$ *and* $T(B^iC) = B^i(TC)$.

(iv) *For all* $i \geq i_0$, *the following sequences are exact:*

$$0 \to Z^iC \to C^i \to B^{i+1}C \to 0$$
$$0 \to B^iC \to Z^iC \to H^iC \to 0.$$

Proof. In (i),(ii),(iii) the V-flatness follows from the conditions on the tangent space, $T(Z^iC) = Z^i(TC), T(H^iC) = H^i(TC), T(B^iC) = B^i(TC)$, and (1).

We prove (i),(ii),(iii) by descending induction. Suppose $j > i_0$ is such that for all $i \geq j$ the statements in (i),(ii),(iii) hold. Such a j exists because C is bounded from above.

Consider $C^{j-1} \to B^j$. Lemma 3.2.2 implies that $Z^{j-1}C := \ker(C^{j-1} \to B^j)$ exists as a V-reduced module and $\iota(Z^{j-1}C) = Z^{j-1}(\iota(C))$. From the exact sequence

$$0 \to TZ^{j-1}C \to TC^{j-1} \to TB^j \to 0$$

(Lemma 3.2.2) we conclude that $TZ^{j-1}C = Z^{j-1}(TC)$. Clearly, $\ker(C^{j-1} \to B^j) = \ker(C^{j-1} \to C^j)$ so that (i) holds for $j-1$.

Consider $C^{j-2} \to Z^{j-1}C$, we get two exact sequences

$$\iota(C^{j-2}) \to \iota(Z^{j-1}C) \to H^{j-1}(\iota(C)) \to 0 \tag{3.2.3}$$

$$TC^{j-2} \to TZ^{j-1}C \to H^{j-1}(TC) \to 0. \tag{3.2.4}$$

In order to apply Lemma 3.2.3 we observe that (3.2.3) and (3.2.4) imply

$$H^{j-1}(\iota(C))/\mathrm{image}(Z^{j-1}C^2) \cong H^{j-1}(TC),$$

and $H^{j-1}(TC)$ is R-flat by (1) and thus $\mathbb{Z}$-flat by (2). This proves (ii) for $j-1$. Finally, (iii) for $j-1$ follows from Lemma 3.2.2 again.

Statement (iv) follows immediately from (i),(ii),(iii). □

3.2.5. Recall from Section 2.3.4 that for every V-flat module M we have an exact functor $\Lambda\bar{\otimes}M$. A sequence

$$0 \to F_1 \to F_2 \to F_3 \to 0$$

of functors F_i : (nilpotent R-algebras) $\to$ (abelian groups), is called *exact* if it induces an exact sequence of abelian groups after evaluation at every nilpotent algebra.

Lemma 3.2.6. *Let*

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of V*-flat modules. Then*

$$0 \to \Lambda\bar{\otimes}M_1 \to \Lambda\bar{\otimes}M_2 \to \Lambda\bar{\otimes}M_3 \to 0$$

is an exact sequence.

Proof. We need to show that for all nilpotent R-algebras $\mathcal{N}$ the sequence

$$0 \to \Lambda(\mathcal{N})\bar{\otimes}M_1 \to \Lambda(\mathcal{N})\bar{\otimes}M_2 \to \Lambda(\mathcal{N})\bar{\otimes}M_3 \to 0 \tag{3.2.5}$$

is exact. Exactness at $\Lambda(\mathcal{N})\bar{\otimes}M_3$ is obvious. At $\Lambda(\mathcal{N})\bar{\otimes}M_2$ we need to show that

$$(\Lambda(\mathcal{N}) \otimes_{\mathbb{E}_R} M_2)_\infty \to (\Lambda(\mathcal{N}) \otimes_{\mathbb{E}_R} M_3)_\infty$$

is surjective. This follows, because $M_2^n \to M_3^n$ and thus $\Lambda(\mathcal{N})_n \otimes_{\mathbb{Z}} M_2^n \to \Lambda(\mathcal{N})_n \otimes_{\mathbb{Z}} M_3^n$ is surjective [Zin84, 3.24(ii)].

Finally, let us prove that

$$\Lambda(\mathcal{N})\bar{\otimes}M_1 \to \Lambda(\mathcal{N})\bar{\otimes}M_2 \quad \text{is injective.} \tag{3.2.6}$$

Given the exactness of the functors $\Lambda\bar{\otimes}M_i$, the statement (3.2.6) for $\mathcal{N}$ follows from the statement (3.2.6) for $\mathcal{N}'$ and $\mathcal{N}''$ if

$$0 \to \mathcal{N}' \to \mathcal{N} \to \mathcal{N}'' \to 0$$

is an exact sequence. By definition there exists an integer n with $\mathcal{N}^n = 0$, thus we may reduce to the case $n = 2$. If $\mathcal{N}^2 = 0$ then

$$\Lambda(\mathcal{N})\bar{\otimes}M_i \cong \mathcal{N} \otimes_R T(M_i)$$

[Zin84, 3.23(c)]. By Lemma 3.2.2 the sequence

$$0 \to T(M_1) \to T(M_2) \to T(M_3) \to 0$$

is exact. Since $T(M_3)$ is a flat R-module the sequence remains exact after tensoring with $\mathcal{N}$. □

3.3. **Proof of the main theorem.**

Proof of Theorem 3.1.1. In view of the uniqueness of a pro-representing formal group we may assume that $S = \mathrm{Spec}(R)$ is affine. Moreover we may assume that X is connected.

By Proposition 1.2.7 it suffices to show that $\Psi^q(X/S, E)$ is pro-representable by a commutative formal Lie group (see (1.2.4) for the definition of $\Psi^q(X/S, E)$).

Let $\{U_i\}_{i=1,\dots,n}$ be an affine covering of X. We set $A_J = \Gamma(\bigcap_{k=0}^{s} U_{\jmath_k}, \mathcal{O}_X)$ for all $J = (\jmath_0 < \cdots < \jmath_s)$. We may suppose that $E\mid_{U_i} \cong \mathrm{Spf}(A_i[|x_1, \dots, x_d|])$ as formal schemes over A_i.

If $\mathcal{N}$ is a nilpotent R-algebra then Lemma 1.1.9 implies that

$$H^q(X, \tilde{E}_{\mathcal{N}}) = H^q(\mathcal{C}^*(\{U_i\}, \tilde{E}_{\mathcal{N}})),$$

where $\mathcal{C}^*(\{U_i\}, \tilde{E}_{\mathcal{N}})$ is the associated Čech complex. Explicitly, we have

$$\mathcal{C}^p(\{U_i\}, \tilde{E}_{\mathcal{N}}) = \bigoplus_{|J|=p+1} F_{A_J,E}(\mathcal{N}),$$

where the functors $F_{A_J,E}$ have been introduced in Section 2.3.11. We claim that the complex of Cartier modules

$$C := \mathcal{C}^*(\{U_i\}, \tilde{E}_{xR[|x|]})$$

satisfies the assumptions of Lemma 3.2.4. The components of C are V-flat because $F_{A_J,E}$ is exact and commutes with arbitrary direct sums (and using Theorem 2.3.8). For the tangent spaces we get

$$T(F_{A_J,E}(xR[|x|])) = F_{A_J,E}(xR[|x|]/x^2R[|x|]) = TE(\cap_k U_{\jmath_k}).$$

Therefore we obtain

$$H^q(T\mathcal{C}^*(\{U_i\}, \tilde{E}_{xR[|x|]})) = H^q(X, TE),$$

which is a locally free R-module for all $q \geq q_0$ by assumption. Thus we may use Lemma 3.2.4 and Lemma 3.2.6 to conclude that

$$\Psi^q(X/S, E) = \Lambda\bar{\otimes}H^qC,$$

with H^qC being a V-flat module, and

$$T\Psi^q(X/S, E) = TH^qC = H^q(X, TE). \tag{3.3.1}$$

Therefore $\Psi^q(X/S,E)$ is exact and commutes with arbitrary direct sums (Theorem 2.3.8). After shrinking $\mathrm{Spec}(R)$ we may also assume that $T\Psi^q(X/S,E)$ is free and finitely generated. Thus Proposition 2.3.10 implies that $\Psi^q(X/S,E)$, and hence $\Phi^q(X/S,E)$, is pro-representable by a commutative formal Lie group $\mathfrak{X}$.

Finally, by definition we have

$$T\Psi^q(X/S,E) = \ker(\mathfrak{X}(\mathrm{Spec}(R[x]/x^2)) \to \mathfrak{X}(\mathrm{Spec}(R))) = T\mathfrak{X}(\mathrm{Spec}(R)),$$

and (3.3.1) induces a natural isomorphism

$$H^q\widetilde{(X,TE)} \cong T\mathfrak{X}.$$

For a not necessarily affine scheme S, the isomorphisms glue and yield

$$R^q f_* TE \cong T\Phi^q(X/S,E).$$

□

Proposition 3.3.1. *Let $f : X \to S$ be a flat separated morphism between noetherian schemes. Suppose that $S = \mathrm{Spec}(R)$ is flat over $\mathrm{Spec}(\mathbb{Z})$. Suppose that $R^q f_*\mathcal{O}_X$ is locally free and of finite rank for all $q \geq q_0$. The Cartier module of $\Phi^{q_0}(X/S,\hat{\mathbb{G}}_m)$ is given by $H^{q_0}(X,\mathbb{W}\mathcal{O}_X)$ where $\mathbb{W}\mathcal{O}_X$ is the sheaf of big Witt vectors.*

Proof. With the notations as in the proof of Theorem 3.1.1, the Cartier module of $\Phi^{q_0}(X/S,\hat{\mathbb{G}}_m)$ equals $H^{q_0}C$. The Cartier module of $F_{A_J,\hat{\mathbb{G}}_m}$ is $\mathbb{W}A_J$ and the cohomology of $\mathbb{W}\mathcal{O}_X$ can be computed via the Čech cohomology for a finite affine covering. □

4. Applications and examples

4.1. One dimensional formal groups over $\mathbb{Z}[N^{-1}]$.

4.1.1. *Setting.* Let $N \geq 1$ be an integer. Let $f : X \to \mathrm{Spec}(\mathbb{Z}[N^{-1}])$ be a flat projective morphism. Suppose that $R^q f_*\mathcal{O}_X$ is free for all $q \geq q_0$. In view of Theorem 3.1.1, $\Phi^{q_0}(X/\mathrm{Spec}(\mathbb{Z}[N^{-1}]),\hat{\mathbb{G}}_m)$ is pro-representable by a commutative formal Lie group over $\mathbb{Z}[N^{-1}]$. In the following we will be interested in the case where $\Phi^{q_0}(X/\mathrm{Spec}(\mathbb{Z}[N^{-1}]),\hat{\mathbb{G}}_m)$ is one-dimensional. For this we need to recall results by Honda on the relation between one-dimensional formal groups and Dirichlet series [Hon70].

4.1.2. Let $\mathfrak{X}$ be a one-dimensional commutative formal Lie group over $\mathbb{Z}[N^{-1}]$. For a $\mathbb{Z}[N^{-1}]$-algebra R we denote by $\mathfrak{X}_R$ the base change to R. Let $\phi : T\mathfrak{X} \to \mathbb{Z}[N^{-1}]$ be an isomorphism. Then there is a unique isomorphism

$$\log_\phi : \mathfrak{X}_\mathbb{Q} \to \hat{\mathbb{G}}_{a,\mathbb{Q}}$$

such that $T\log_\phi = \phi \otimes_\mathbb{Z} \mathbb{Q}$ (see A.2). Therefore we can form the composition

$$\mathfrak{X}(\mathbb{Z}[N^{-1}][|x|]) \to \mathfrak{X}_\mathbb{Q}(\mathbb{Q}[|x|]) \xrightarrow{\log_\phi} \hat{\mathbb{G}}_{a,\mathbb{Q}}(\mathbb{Q}[|x|]) = x\mathbb{Q}[|x|]. \tag{4.1.1}$$

The image of $\mathfrak{X}(\mathbb{Z}[N^{-1}][|x|])$ in $x\mathbb{Q}[|x|]$, which will be denoted by $\log(\mathfrak{X})$ in the following, depends only on the isomorphism class of $\mathfrak{X}$. Indeed, if $\tau : \mathfrak{X}' \to \mathfrak{X}$ and $\phi' : T\mathfrak{X}' \to \mathbb{Z}[N^{-1}]$ are some isomorphisms then $\phi \circ T\tau = \lambda\cdot\phi'$ for some $\lambda \in \mathbb{Z}[N^{-1}]^*$. But multiplication with $N : \mathfrak{X}' \to \mathfrak{X}'$ is an isomorphism and therefore there exists an automorphism $[\lambda] : \mathfrak{X}' \to \mathfrak{X}'$ such that $T[\lambda]$ is multiplication by λ. We obtain $\log_\phi \circ\tau = \log_{\phi'} \circ[\lambda]$.

4.1.3. Suppose $f \in \log(\mathfrak{X})$ and $f = x \bmod x^2$. Let f^{-1} be the inverse of f, i.e. $f^{-1}(f(x)) = x$. Then $\mathfrak{X}$ is isomorphic to the group $\mathrm{Spf}(\mathbb{Z}[N^{-1}][|x|])$ with multiplication

$$m^*(x) = f^{-1}(f(x)\hat{\otimes}1 + 1\hat{\otimes}f(x)).$$

A fundamental example is $\mathfrak{X} = \hat{\mathbb{G}}_m$ and $f = \sum_{k=1}^{\infty} \frac{x^k}{k} \in \log(\hat{\mathbb{G}}_m)$ (with the sign convention of Section 1.1.1).

4.1.4. We define a subset $\mathfrak{Di}_N$ of $x\mathbb{Q}[|x|]$ as follows. A series $\sum_{m=1}^{\infty} \frac{A_m}{m} x^m$ belongs to $\mathfrak{Di}_N$ if and only if for all primes $p \nmid N$ and all integers $n \geq 1$ there exists an integer C_{p^n} such that

$$\sum_{m=1}^{\infty} A_m m^{-s} = \prod_{p \nmid N} \frac{1}{1 + \sum_{n=1}^{\infty} p^{n-1} C_{p^n} p^{-ns}}.$$

(cf. [Hon70, §6.1, p.239]). Given $f \in \mathfrak{Di}_N$, the integers C_{p^n} are unique. Obviously, $f \in \mathfrak{Di}_N$ implies $f = x \bmod x^2$.

Definition 4.1.5. For $\mathfrak{X}$ as in Section 4.1.2 we set

$$\mathfrak{Di}_N(\mathfrak{X}) := \mathfrak{Di}_N \cap \log(\mathfrak{X}).$$

Theorem 4.1.6 (Honda)**.** *Let $\mathfrak{X}$ be as in Section 4.1.2.*

(1) *The set $\mathfrak{Di}_N(\mathfrak{X})$ is non-empty.*
(2) *Suppose $f \in \mathfrak{Di}_N(\mathfrak{X})$. Then $\tilde{f} \in \mathfrak{Di}_N(\mathfrak{X})$ if and only if the following conditions hold:*
   (a) *$\tilde{f} \in \mathfrak{Di}_N$, we write $\tilde{f} = \sum_{m=1}^{\infty} \frac{\tilde{A}_m}{m} x^m$ and*

$$\sum_{m=1}^{\infty} \tilde{A}_m m^{-s} = \prod_{p \nmid N} \frac{1}{1 + \sum_{n \geq 1} p^{n-1} \tilde{C}_{p^n} p^{-ns}}.$$

   (b) *For all primes $p \nmid N$, we have an equality of ideals*

$$\mathbb{Z}_p[|x|] \cdot (p + \sum_{n=1}^{\infty} C_{p^n} x^n) = \mathbb{Z}_p[|x|] \cdot (p + \sum_{n=1}^{\infty} \tilde{C}_{p^n} x^n)$$

   in $\mathbb{Z}_p[|x|]$.

Proof. For (1). [Hon70, §6, Corollary 1, p.239] The proof in loc. cit. is for $N = 1$ but works in the same way for $N > 1$.

For (2). Suppose first that $\tilde{f} \in \mathfrak{Di}_N(\mathfrak{X})$. Let $p \nmid N$ be a prime. By [Hon70, Theorem 8, p.239] we know that

$$pf(x) + \sum_{n=1}^{\infty} C_{p^n} f(x^{p^n}) = 0 \mod p\mathbb{Z}_p,$$

$$p\tilde{f}(x) + \sum_{n=1}^{\infty} \tilde{C}_{p^n} \tilde{f}(x^{p^n}) = 0 \mod p\mathbb{Z}_p.$$

Since $\tilde{f} \in \mathfrak{Di}_N(\mathfrak{X})$ we obtain $f^{-1} \circ \tilde{f} \in x\mathbb{Z}[N^{-1}][|x|]$. By [Hon70, §2, Theorem 3, p.225] this implies (b).

Suppose now that $\tilde{f} \in \mathfrak{Di}_N$ and (b) holds. We consider $f^{-1} \circ \tilde{f} \in x\mathbb{Q}[|x|]$. It is sufficient to show that $f^{-1} \circ \tilde{f} \in x\mathbb{Z}_p[|x|]$ for all primes $p \nmid N$, because it implies $f^{-1} \circ \tilde{f} \in x\mathbb{Z}[N^{-1}][|x|]$ and $\tilde{f} \in \log(\mathfrak{X})$.

Again,

$$p\tilde{f}(x) + \sum_{n=1}^{\infty} \tilde{C}_{p^n} \tilde{f}(x^{p^n}) = 0 \mod p\mathbb{Z}_p,$$

and since $p + \sum_{n=1}^{\infty} C_{p^n} x^n = t \cdot (p + \sum_{n=1}^{\infty} \tilde{C}_{p^n} x^n)$ for a unit $t \in \mathbb{Z}_p[|x|]$, we obtain

$$p\tilde{f}(x) + \sum_{n=1}^{\infty} C_{p^n} \tilde{f}(x^{p^n}) = 0 \mod p\mathbb{Z}_p.$$

In view of [Hon70, Theorem 2, p.223] this implies $f^{-1} \circ \tilde{f} \in x\mathbb{Z}_p[|x|]$. □

Remark 4.1.7. Honda also proves that for every $f \in \mathfrak{Di}_N$ there is a one-dimensional commutative formal Lie group $\mathfrak{X}$ over $\mathbb{Z}[N^{-1}]$ such that $f \in \mathfrak{Di}_N(\mathfrak{X})$ [Hon70, §6, Corollary 1, p.239].

4.1.8. Suppose that $f \in \mathfrak{Di}_N(\mathfrak{X})$, with $f = \sum_{m=1}^{\infty} \frac{A_m}{m} x^m$ and

$$\sum_{m=1}^{\infty} A_m m^{-s} = \prod_{p \nmid N} \frac{1}{1 + \sum_{n=1}^{\infty} p^{n-1} C_{p^n} p^{-ns}}.$$

From Theorem 4.1.6 we conclude that the ideal

$$\mathbb{Z}_p[|x|] \cdot (p + \sum_{n=1}^{\infty} C_{p^n} x^n) \subset \mathbb{Z}_p[|x|] \tag{4.1.2}$$

is an invariant of $\mathfrak{X}$. In fact it depends only on the base change $\mathfrak{X}_{\mathbb{Z}_p}$ together with $\phi \otimes_{\mathbb{Z}[N^{-1}]} \mathbb{Z}_p$, where $\phi : T\mathfrak{X} \to \mathbb{Z}[N^{-1}]$ is an isomorphism. By [Hon70, Theorem 4, p.228] the isomorphism classes of pairs $(\mathfrak{Y}, \psi)$, with $\mathfrak{Y}$ a commutative one-dimensional formal Lie group over $\mathbb{Z}_p$ and $\psi : T\mathfrak{Y} \to \mathbb{Z}_p$ an isomorphism, correspond to ideals of the form (4.1.2). The correspondence works as follows.

For $(\mathfrak{Y}, \psi)$ the corresponding ideal $(p + \sum_{n=1}^{\infty} C_{p^n} x^n)$ is defined by the property

$$\begin{aligned} &\log_{\psi}(\mathfrak{Y}(\mathbb{Z}_p[|x|])) \cap \{f \in \mathbb{Q}_p[|x|] \mid f = x \bmod x^2\} \\ &\quad = \{f \in \mathbb{Q}_p[|x|] \mid f = x \bmod x^2, pf(x) + \sum_{n=1}^{\infty} C_{p^n} f(x^{p^n}) = 0 \bmod p\mathbb{Z}_p\}. \end{aligned} \tag{4.1.3}$$

Equivalently, we may require

$$pf(x) + \sum_{n=1}^{\infty} C_{p^n} f(x^{p^n}) = 0 \mod p\mathbb{Z}_p$$

for only one element $f \in \log_{\psi}(\mathfrak{Y}(\mathbb{Z}_p[|x|])) \cap \{f \in \mathbb{Q}_p[|x|] \mid f = x \bmod x^2\}$.

4.1.9. The ideal $(p + \sum_{n=1}^{\infty} C_{p^n} x^n)$ attached to $(\mathfrak{Y}, \psi)$ also appears in the following way. Let $\mathfrak{Y}_{\mathbb{F}_p}$ be the reduction modulo p. The p-typical Cartier module M_p associated to $\mathfrak{Y}_{\mathbb{F}_p}$ is a V-reduced $\mathbb{E}_{\mathbb{F}_p,p}$-module (see Section 2.3.13). The p-typical Cartier ring $\mathbb{E}_{\mathbb{F}_p,p}$ contains $W(\mathbb{F}_p)[|V|]$, which we may identify with $\mathbb{Z}_p[|x|]$ via $W(\mathbb{F}_p) = \mathbb{Z}_p$ and $V \leftrightarrow x$. We denote by $\mathbb{Q}_p\{V\}$ the ring $\mathbb{Z}_p[|V|] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$.

Lemma 4.1.10. *There is an isomorphism of* $\mathbb{Q}_p\{V\}$*-modules*

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} M_p \cong \mathbb{Q}_p\{V\}/(p + \sum_{n=1}^{\infty} C_{p^n} V^n).$$

Proof. Define $\tilde{M} := \mathbb{E}_{\mathbb{Z}_p}/I$ where the left ideal I is generated by $F_p+\sum_{n=1}^{\infty} C_{p^n} V_p^{n-1}$ and F_ℓ for $\ell \neq p$ (we have to remark that there exists a unique ring homomorphism $\rho : \mathbb{Z}_p \to \mathbb{W}(\mathbb{Z}_p)$ such that $F_n(\rho(a)) = \rho(a)$ for all n and $a \in \mathbb{Z}_p$; we will drop ρ, e.g. $C_{p^n} = \rho(C_{p^n})$).

Let $e \in \tilde{M}$ be the image of 1. It is easy to see that $\tilde{M}$ together with the filtration $\tilde{M}^n = \text{image}(\mathbb{E}_{\mathbb{Z}_p,n})$ is a V-flat module and there is a unique isomorphism $\phi : \tilde{M}/\tilde{M}^2 \to \mathbb{Z}_p$ with $\phi^{-1}(1) = e$. Let $\mathfrak{X}$ be the one-dimensional formal Lie group over $\mathbb{Z}_p$ attached to $\tilde{M}$ via Cartier theory. Then e defines a curve $\mathfrak{X}(\mathbb{Z}_p[|x|])$, and ϕ induces an isomorphism $\phi : T\mathfrak{X} \to \mathbb{Z}_p$.

Set $f := \log_\phi(e) \in \hat{\mathbb{G}}_a(\mathbb{Q}_p[|x|]) = x\mathbb{Q}_p[|x|]$; by construction $f = x \bmod x^2$. Recall from (2.2.3) the F_n and V_n operation on $\hat{\mathbb{G}}_a(\mathbb{Q}_p[|x|])$. From $F_\ell(e) = 0$ for $\ell \neq p$ we obtain

$$f = x + \sum_{n=1}^{\infty} a_n x^{p^n},$$

for some $a_n \in \mathbb{Q}_p$. Now,

$$\begin{aligned}(p + \sum_{n=1}^{\infty} C_{p^n} V_p^n)(f) &= (V_p(F_p + \sum_{n=1}^{\infty} C_{p^n} V_p^{n-1}))(f) - (V_pF_p - p)(f) \\ &= -(V_pF_p - p)(f) \\ &= px.\end{aligned}$$

Therefore $p+\sum_{n=1}^{\infty} C_{p^n} x^n$ is attached to $(\mathfrak{X}, \phi)$ via Honda's correspondence (4.1.3).

The p-typical Cartier module M_p of $\mathfrak{X}_{\mathbb{F}_p}$ is simply $\mathbb{E}_{\mathbb{F}_p,p}/(F + \sum_{n=1}^{\infty} C_{p^n} V^{n-1})$ and

$$\begin{gathered}\mathbb{Q}_p\{V\}/(p + \sum_{n=1}^{\infty} C_{p^n} V^n) \to \mathbb{Q}_p \otimes_{Z_p} \mathbb{E}_{\mathbb{F}_p,p}/(F + \sum_{n=1}^{\infty} C_{p^n} V^{n-1}) \\ 1 \mapsto Ve\end{gathered}$$

is an isomorphism. □

We note that $\mathbb{Q}_p\{V\}^* = \mathbb{Q}_p^* \cdot \mathbb{Z}_p[|V|]^*$ and thus Lemma 4.1.10 implies that the ideal $(p+\sum_{n=1}^{\infty} C_{p^n} x^n)$ attached to $(\mathfrak{Y}, \psi)$ depends only on the isomorphism class of $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} M_p$ as $\mathbb{Q}_p\{V\}$-module, where M_p is the p-typical Cartier module associated to $\mathfrak{Y}_{\mathbb{F}_p}$.

Proposition 4.1.11. *Let $f : X \to \mathrm{Spec}(\mathbb{Z}[N^{-1}])$ be a flat projective morphism. Suppose that $R^q f_* \mathcal{O}_X$ is free for all $q \geq q_0$. Suppose $\mathfrak{X} = \Phi^{q_0}(X/\mathrm{Spec}(\mathbb{Z}[N^{-1}]), \hat{\mathbb{G}}_m)$ is a one-dimensional. Let $f \in \mathfrak{Di}_N$ and write $f = \sum_{m=1}^{\infty} \frac{A_m}{m} x^m$ where*

$$\sum_{m=1}^{\infty} A_m m^{-s} = \prod_{p \nmid N} \frac{1}{1 + \sum_{n=1}^{\infty} p^{n-1} C_{p^n} p^{-ns}}$$

(see Section 4.1.4). The following statements are equivalent:

(i) $f \in \mathfrak{Di}_N(\mathfrak{X})$.

(ii) *For every prime $p \nmid N$ there is an isomorphism*

$$\mathbb{Q}_p\{V\}/(p + \sum_{n=1}^{\infty} C_{p^n} V^n) \to \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^{q_0}(X \otimes_{\mathbb{Z}} \mathbb{F}_p, W\mathcal{O}_{X \otimes_{\mathbb{Z}} \mathbb{F}_p})$$

as $\mathbb{Q}_p\{V\}$-modules.

(iii) *For every prime $p \nmid N$ there is an isomorphism*

$$\mathbb{Q}_p\{V\}/(p + \sum_{n=1}^{\infty} C_{p^n} V^n) \to H^{q_0}_{\mathrm{rig}}(X \otimes_{\mathbb{Z}} \mathbb{F}_p/\mathbb{Q}_p)_{<1}$$

as $\mathbb{Q}_p\{V\}$-modules.

Proof. The equivalence of (ii) and (iii) is implied by [BBE07, Theorem 1.1].

In view of Theorem 3.1.1, $\Phi^{q_0}(X/\mathrm{Spec}(\mathbb{Z}[N^{-1}]), \hat{\mathbb{G}}_m)$ is pro-representable by a one-dimensional formal Lie group $\mathfrak{X}$. Thus $\Phi^{q_0}(X \otimes_{\mathbb{Z}} \mathbb{F}_p/\mathrm{Spec}(\mathbb{F}_p), \hat{\mathbb{G}}_m)$ is pro-representable by the base change $\mathfrak{X}_{\mathbb{F}_p}$.

By [AM77, Corollary 4.3] we know that $H^{q_0}(X \otimes_{\mathbb{Z}} \mathbb{F}_p, W\mathcal{O}_{X \otimes_{\mathbb{Z}} \mathbb{F}_p})$ is the p-typical Cartier module associated to $\mathfrak{X}_{\mathbb{F}_p}$. Therefore Lemma 4.1.10 implies the equivalence. □

4.2. **Gauss-Manin connection.**

4.2.1. Let S be a smooth scheme over $\mathrm{Spec}(\mathbb{Z})$. Let $\mathfrak{X}$ be a commutative formal Lie group over S. We have the de-Rham complex $\Omega^*_{\mathfrak{X}/S}$ at disposal and the Katz-Oda [KO68] construction yields an integrable connection

$$\nabla : H^i(\Omega^*_{\mathfrak{X}/S}) \to H^i(\Omega^*_{\mathfrak{X}/S}) \otimes_{\mathcal{O}_S} \Omega^1_{S/\mathbb{Z}} \qquad \text{for all } i.$$

Locally, we can write a closed i-form ω as

$$\omega = \sum_{J=(j_1,\ldots,j_i)} \omega_J \cdot dx_{j_1} \wedge \cdots \wedge dx_{j_i},$$

with $\omega_J = \sum_{K=(k_1,\ldots,k_d)} a_{J,K} \cdot x_1^{k_1} \cdots x_d^{k_d}$, and $a_{J,K} \in \mathcal{O}_S$. For a section $\xi \in T_{S/\mathbb{Z}}$ the action is simply given by

$$\nabla_\xi(\omega) = \sum_{J=(j_1,\ldots,j_i)} \sum_{K=(k_1,\ldots,k_d)} \xi(a_{J,K}) \cdot x_1^{k_1} \cdots x_d^{k_d} \cdot dx_{j_1} \wedge \cdots \wedge dx_{j_i}.$$

In view of Proposition A.1.3 we have a morphism

$$\eta : T\mathfrak{X}^\vee \to \Omega^1_{\mathfrak{X}/S},$$

inducing an isomorphism with the invariant 1-forms. By using Lemma A.1.4 we obtain

$$\bar{\eta} : T\mathfrak{X}^\vee \to H^1(\Omega^*_{\mathfrak{X}/S}). \tag{4.2.1}$$

Definition 4.2.2. We define $H^1_{\mathrm{inv}}(\mathfrak{X}) \subset H^1(\Omega^*_{\mathfrak{X}/S})$ to be the smallest $\mathcal{O}_S$-submodule such that $H^1_{\mathrm{inv}}(\mathfrak{X})$ contains the image of $\bar{\eta}$, and the connection on $H^1(\Omega^*_{\mathfrak{X}/S})$ induces a connection on $H^1_{\mathrm{inv}}(\mathfrak{X})$.

4.2.3. It is more convenient to work with modules rather than with connections. We denote by D_S the sheaf differential operators and we denote by D'_S the subsheaf generated (as $\mathcal{O}_S$-algebra) by differential operators of order ≤ 1. Locally, we can write $S = \mathrm{Spec}(R)$ and suppose that there is an étale morphism $\mathbb{Z}[x_1, \ldots, x_n] \to R$. As an R-module we have

$$D'_S(S) = R \otimes_{\mathbb{Z}} \mathbb{Z}[\partial_1, \ldots, \partial_n]$$

with $\partial_i(x_j) = \delta_{i,j}$, the ring structure beeing uniquely determined by

$$\partial_i \partial_j = \partial_j \partial_i, \quad \partial_i r = r\partial_i + \partial_i(r) \quad \text{for all } i, j \text{ and } r \in R.$$

We have an obvious equivalence of categories between $\mathcal{O}_S$-modules with integrable connection and D'_S-modules. By definition, $H^1_{\mathrm{inv}}(\mathfrak{X})$ is the D'_S-submodule of $H^1(\Omega^*_{\mathfrak{X}/S})$ generated by $\bar{\eta}(T\mathfrak{X}^\vee)$, thus it comes equipped with a surjective morphism

$$D'_S \otimes_{\mathcal{O}_S} T\mathfrak{X}^\vee \to H^1_{\mathrm{inv}}(\mathfrak{X}).$$

In particular, $H^1_{\mathrm{inv}}(\mathfrak{X})$ is a quasi-coherent $\mathcal{O}_S$-module.

4.2.4. Let $f : X \to S$ be a smooth, projective morphism of relative dimension r. As above we assume that S is smooth over $\mathrm{Spec}(\mathbb{Z})$. The de-Rham cohomology

$$H^i_{\mathrm{dR}}(X/S) := R^i f_*(\Omega^*_{X/S})$$

comes equipped with the Gauss-Manin connection, hence defines a D'_S-module.

Theorem 4.2.5. *Let $f : X \to S$ be a smooth, projective morphism of relative dimension r. Suppose that S is smooth over* $\mathrm{Spec}(\mathbb{Z})$ *and suppose that $R^j f_*\Omega^i_{X/S}$ is locally free for all i, j. Then $H^1_{\mathrm{inv}}(\Phi^i(X/S))$ is a subquotient of $H^{2r-i}_{\mathrm{dR}}(X/S)$ as D'_S-module for all $i \geq 0$.*

In fact, it will be easier to prove the theorem when stated in a more precise way (Theorem 4.2.7). Theorem 3.1.1 guarantees the pro-representability of $\Phi^i(X/S) = \Phi^i(X/S, \hat{\mathbb{G}}_m)$. Let $X_{\mathbb{Q}}$ and $S_{\mathbb{Q}}$ denote the base change to $\mathrm{Spec}(\mathbb{Q})$. Obviously, we have

$$H^{2r-i}_{\mathrm{dR}}(X/S) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_{\mathbb{Q}}} = H^i_{\mathrm{dR}}(X_{\mathbb{Q}}/S_{\mathbb{Q}}).$$

Hence, Theorem 4.2.5 implies the following corollary.

Corollary 4.2.6. *Let $f : X \to S$ be a smooth, projective morphism of relative dimension r. Suppose that S is smooth over* $\mathrm{Spec}(\mathbb{Z})$ *and suppose that $R^j f_*\mathcal{O}_X$ is locally free for all j. Then, for all i, the sheaf $H^1_{\mathrm{inv}}(\Phi^i(X/S)) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_{\mathbb{Q}}}$ is a subquotient of $H^{2r-i}_{\mathrm{dR}}(X_{\mathbb{Q}}/S_{\mathbb{Q}})$ as module over the sheaf of differential operators of $S_{\mathbb{Q}}$.*

Proof. We know that $R^j f_*\Omega^i_{X_{\mathbb{Q}}/S_{\mathbb{Q}}}$ is locally free by Hodge theory. Thus there exists an integer $n \neq 0$ such that the base change of f to $S_{\mathbb{Z}[n^{-1}]} = S \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Z}[n^{-1}])$ satisfies the assumptions in Theorem 4.2.5. Since

$$H^1_{\mathrm{inv}}(\Phi^i(X_{\mathbb{Z}[n^{-1}]}/S_{\mathbb{Z}[n^{-1}]})) \otimes_{\mathcal{O}_{S_{\mathbb{Z}[n^{-1}]}}} \mathcal{O}_{S_{\mathbb{Q}}} = H^1_{\mathrm{inv}}(\Phi^i(X/S)) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_{\mathbb{Q}}},$$

the statement follows. □

Of course, $H^1_{\mathrm{inv}}(\Phi^i(X_{\mathbb{Q}}/S_{\mathbb{Q}}))$ vanishes. But $H^1_{\mathrm{inv}}(\Phi^i(X/S)) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_{\mathbb{Q}}}$ is in general non-trivial (see Examples 4.2.13).

Since f is of relative dimension r, we get a natural morphism

$$\epsilon : R^{r-i} f_*\Omega^r_{X/S} \to H^{2r-i}_{\mathrm{dR}}(X/S).$$

Therefore, $D'_S \cdot \epsilon(R^{r-i} f_*\Omega^r_{X/S})$ is a D'_S-submodule of $H^{2r-i}_{\mathrm{dR}}(X/S)$. Moreover, Grothendieck duality implies the existence of a canonical morphism

$$\tau : R^{r-i} f_*\Omega^r_{X/S} \to (R^i f_*\mathcal{O}_X)^\vee, \tag{4.2.2}$$

coming from the pairing

$$R^{r-i} f_*\Omega^r_{X/S} \times R^i f_*\mathcal{O}_X \to R^r f_*\Omega^r_{X/S} \xrightarrow{\mathrm{Tr}} \mathcal{O}_S. \tag{4.2.3}$$

Let us now give the more precise formulation of Theorem 4.2.5.

Theorem 4.2.7. *Assumptions as in Theorem 4.2.5. For all i, the morphism*

$$R^{r-i}f_*\Omega^r_{X/S} \xrightarrow{(4.2.2)} (R^if_*\mathcal{O}_X)^\vee \xrightarrow{(4.2.1)} H^1_{\mathrm{inv}}(\Phi^i(X/S)) \tag{4.2.4}$$

extends to a surjective morphism of D'_S-modules

$$D'_S \cdot \epsilon(R^{r-i}f_*\Omega^r_{X/S}) \to H^1_{\mathrm{inv}}(\Phi^i(X/S)). \tag{4.2.5}$$

Proof. Note that $R^if_*\mathcal{O}_X = T\Phi^i(X/S)$, so that (4.2.1) makes sense. We may suppose that $S = \mathrm{Spec}(R)$ is affine. Furthermore, we may suppose $\Phi^i(X/S) = \mathrm{Spf}(R[|x_1,\dots,x_d|])$ as formal schemes; in particular we already have a trivialization

$$T\Phi^i(X/S) = \bigoplus_{j=1}^d \mathcal{O}_S e_j. \tag{4.2.6}$$

Denote by $e_1^\vee,\dots,e_d^\vee$ the dual basis, set

$$\alpha_j^\vee := \tau^{-1}(e_j^\vee) \in \Gamma(R^{r-i}f_*\Omega^r_{X/S}),$$
$$\omega_j := \eta(e_j^\vee) \in \Gamma(\Omega^1_{\Phi^i(X/S)/S}), \qquad \bar\omega_j := \bar\eta(e_j^\vee) \in \Gamma(H^1(\Omega^*_{\Phi^i(X/S)/S})).$$

If $P_1,\dots,P_d \in \Gamma(D'_S)$ satisfy

$$\sum_{j=1}^d P_j \cdot \epsilon(\alpha_j^\vee) = 0 \quad \text{in } \Gamma(H^i_{\mathrm{dR}}(X/S)),$$

then we need to show that

$$\sum_{j=1}^d P_j \cdot \bar\omega_j = 0 \quad \text{in } \Gamma(H^1(\Omega^*_{\Phi^i(X/S)/S})). \tag{4.2.7}$$

Our trivialization (4.2.6) induces an isomorphism

$$\log : \Phi^i(X/S) \times_S S_{\mathbb{Q}} \xrightarrow{\cong} \hat{\mathbb{G}}^d_{a,S_{\mathbb{Q}}},$$

where $S_{\mathbb{Q}}$ is the base change to $\mathrm{Spec}(\mathbb{Q})$ (see A.2). Moreover, we have $d\log^*(x_j) = \omega_j$ (Section A.2.3). Write $\log^*(x_j) = \sum_I g_{j,I}x^I$ with $g_{j,I} \in R \otimes_{\mathbb{Z}} \mathbb{Q}$. Then (4.2.7) is equivalent to

$$f := \sum_I \sum_{j=1}^d P_j(g_{j,I})x^I \in R[|x_1,\dots,x_d|].$$

Recall from (A.2.2) that log induces

$$\log : \Phi^i(X/S)(R[|x|]) \to \hat{\mathbb{G}}_a((R\otimes\mathbb{Q})[|x|])^d = \bigoplus_{j=1}^d x\cdot(R\otimes\mathbb{Q})[|x|].$$

Moreover, the following diagram is commutative:

$$\begin{tikzcd} \Phi^i(X/S)(R[|x|]) \arrow[r, "\log"] \arrow[d, "\pi"] & \hat{\mathbb{G}}_a((R\otimes\mathbb{Q})[|x|])^d \arrow[d, "(xf_1,\dots,xf_d)\mapsto(f_1(0),\dots,f_d(0))"] \\ \Gamma(T\Phi^i(X/S)) & \bigoplus_{j=1}^d Re_i, \arrow[l, "(4.2.6)"'] \end{tikzcd}$$

where π is the projection. In view of Proposition 3.3.1 we have

$$\Phi^i(X/S)(R[|x|]) = H^i(X, \mathbb{W}\mathcal{O}_X)$$

and π corresponds to the projection $H^i(X, \mathbb{W}\mathcal{O}_X) \to H^i(X, \mathcal{O}_X)$ induced by the map $\mathbb{W}\mathcal{O}_X \to \mathcal{O}_X$.

Let $\xi \in \Phi^i(X/S)(R[|x|])$ and write $\log(\xi) =: (\log(\xi)_1, \ldots, \log(\xi)_d)$. For $n \geq 1$ we obtain

$$\pi(F_n(\xi)) = \sum_{j=1}^{d} (F_n(\log(\xi)_j))(0) \cdot e_j,$$

because log is a morphism of $\mathbb{E}_R$-modules. Note that $\pi(F_n(\xi)) \in \Gamma(T\Phi^i(X/S))$ implies $(F_n(\log(\xi)_j))(0) \in R$. By construction, $\log(\xi)_j = \xi^* \log^*(x_j) \in (R \otimes \mathbb{Q})[|x|]$, and by (2.2.3) we know that

$$(F_n(\log(\xi)_j))(0) = n \cdot (n\text{-th coefficient of } \xi^* \log^*(x_j)).$$

A theorem of Stienstra 4.2.12 implies

$$\sum_{j=1}^{d} P_j \cdot (F_n(\log(\xi)_j))(0) \in nR,$$

hence

$$\sum_{j=1}^{d} P_j \cdot (n\text{-th coefficient of } \xi^* \log^*(x_j)) \in R \qquad \text{for all } n. \tag{4.2.8}$$

Now suppose that $\xi^*(x_j) \in \mathbb{Z}[|x|]$ for every $j = 1, \ldots, d$ (where the x_j are now the coordinates from $\Phi^i(X/S) = \mathrm{Spf}(R[|x_1, \ldots, x_d|])$). Then (4.2.8) implies that $\xi^*(f) \in R[|x|]$. In view of Lemma 4.2.8 we finally obtain $f \in R[|x_1, \ldots, x_d|]$. $\square$

Lemma 4.2.8. *Let R be flat over $\mathbb{Z}$. Let $f \in (R \otimes_{\mathbb{Z}} \mathbb{Q})[|x_1, \ldots, x_d|]$. Suppose that $\xi^*(f) \in R[|x|]$ for all $\xi : \mathrm{Spf}(R[|x|]) \to \mathrm{Spf}(R[|x_1, \ldots, x_d|])$ such that $\xi^*(x_j) \in \mathbb{Z}[|x|]$ for all j. Then $f \in R[|x_1, \ldots, x_d|]$.*

Proof. We prove by induction on d. The case $d = 1$ is trivial. By subtracting an element in $R[|x_1, \ldots, x_d|]$ we may assume that

$$f = x_d^m \cdot \sum_{i=0}^{\infty} f_i(x_0, \ldots, x_{d-1}) x_d^i$$

and $f_0 \notin R[|x_0, \ldots, x_{d-1}|]$. For every $\xi' : \mathrm{Spf}(R[|x|]) \to \mathrm{Spf}(R[|x_1, \ldots, x_{d-1}|])$ such that $\xi'^*(x_j) \in \mathbb{Z}[|x|]$ for all j, and every integer $n \geq 1$, we define ξ by $\xi^*(x_j) = \xi'^*(x_j)$ if $j < d$ and $\xi^*(x_d) = x_d^n$. Let n tend to ∞ in order to obtain $\xi'^*(f_0) \in R[|x_1, \ldots, x_{d-1}|]$. $\square$

As in Corollary 4.2.6 we obtain the following statement.

Corollary 4.2.9. *Let $f : X \to S$ be a smooth, projective morphism of relative dimension r. Suppose that S is smooth over $\mathrm{Spec}(\mathbb{Z})$ and suppose that $R^j f_* \mathcal{O}_X$ is locally free for all j. Then, for all i, the sheaf $H^1_{\mathrm{inv}}(\Phi^i(X/S)) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_{\mathbb{Q}}}$ is a quotient of the submodule $D_{S_{\mathbb{Q}}} \cdot R^{r-i} f_* \Omega^r_{X_{\mathbb{Q}}/S_{\mathbb{Q}}}$ of $H^{2r-i}_{\mathrm{dR}}(X_{\mathbb{Q}}/S_{\mathbb{Q}})$.*

Remark 4.2.10. We don't know examples where the quotient map

$$D_{S_{\mathbb{Q}}} \cdot R^{r-i} f_* \Omega^r_{X_{\mathbb{Q}}/S_{\mathbb{Q}}} \to H^1_{\mathrm{inv}}(\Phi^i(X/S)) \otimes_{\mathcal{O}_S} \mathcal{O}_{S_{\mathbb{Q}}}$$

is not an isomorphism.

4.2.11. In the following we recall the theorem of Stienstra that is used in the proof of Theorem 4.2.7. Stienstra proved it in [Sti91, Theorem 4.6] by using his definition of the big de Rham-Witt complex. For the convenience of the reader we will recall the proof and work with the big de Rham-Witt complex introduced in [Hes].

Theorem 4.2.12 ([Sti91, Theorem 4.6]). *Let $f : X \to S$ be a smooth, projective morphism of relative dimension r. Suppose that $S = Spec(R)$ is smooth over $\mathrm{Spec}(\mathbb{Z})$ and suppose that $R^j f_* \Omega^i_{X/S}$ is free for all i, j. Fix an integer $m \geq 0$. Take a basis $\omega_1, \dots, \omega_h$ of $H^m(X, \mathcal{O}_X)$. Let $\omega_1^\vee, \dots, \omega_h^\vee$ be the dual basis of $H^{r-m}(X, \Omega^r_{X/S})$. Take $\xi \in H^m(X, \mathbb{W}\mathcal{O}_X)$ and define for every positive integer n the elements $B_{n,1}, \dots, B_{n,h} \in R$ by*

$$\pi(F_n \xi) = \sum_{j=1}^{h} B_{n,j} \omega_j,$$

where $\pi : H^m(X, \mathbb{W}\mathcal{O}_X) \to H^m(X, \mathcal{O}_X)$ is the projection.

Suppose $P_1, \dots, P_h \in \Gamma(D'_S)$ are such that

$$\sum_{j=1}^{h} P_j \omega_j^\vee = 0 \qquad \textit{in } H^{2r-m}(X, \Omega^*_{X/S}),$$

then

$$\sum_{j=1}^{h} P_j B_{n,j} = 0 \mod n.$$

Proof. For a truncation set T, let $\mathbb{W}_T \Omega^*_X$ be the big de-Rham Witt complex defined in [Hes]. We will only need a relative version: set

$$\mathbb{W}\Omega^*_{X/S} := \varprojlim_T \mathbb{W}_T \Omega^*_X / (f^{-1} \mathbb{W}_T \Omega^1_S \cdot \mathbb{W}_T \Omega^*_X),$$

where T runs over all finite truncation sets. We denote by τ the projection $\mathbb{W}\Omega^*_{X/S} \to \Omega^*_{X/S}$, it is a morphism of differential graded algebras. For all $n \geq 1$ we have the Frobenius $F_n : \mathbb{W}\Omega^*_{X/S} \to \mathbb{W}\Omega^*_{X/S}$ satisfying $dF_n = n \cdot F_n d$. Moreover, there is a natural morphism

$$\eta_{X/S} : \mathbb{W}\mathcal{O}_X \to \mathbb{W}\Omega^*_{X/S},$$

that identifies $\mathbb{W}\Omega^0_{X/S}$ with $\mathbb{W}\mathcal{O}_X$, but $\eta_{X/S}$ is in general not a morphism of complexes.

For all integers $n \geq 1$, we set

$$\mathbb{Z}/n \otimes \mathbb{W}\Omega^*_{X/S} := \mathrm{coker}(\mathbb{W}\Omega^*_{X/S} \xrightarrow{n\cdot} \mathbb{W}\Omega^*_{X/S}),$$

and similarly we define $\mathbb{Z}/n \otimes \Omega^*_{X/S}$; we denote by ϵ_n the quotient map $\mathbb{W}\Omega^*_{X/S} \to \mathbb{Z}/n \otimes \mathbb{W}\Omega^*_{X/S}$. The rule $dF_n = nF_n d$ implies that we obtain a commutative diagram of morphisms of complexes

$$\begin{tikzcd}
\mathbb{W}\mathcal{O}_X \arrow[rr, "\epsilon_n \circ \eta_{X/S} \circ F_n"] \arrow[dr, "\epsilon_n \circ \eta_{X/\mathbb{Z}} \circ F_n"'] & & \mathbb{Z}/n \otimes \mathbb{W}\Omega^*_{X/S} \\
& \mathbb{Z}/n \otimes \mathbb{W}\Omega^*_{X/\mathbb{Z}} \arrow[ur] &
\end{tikzcd}$$

We set

$$\zeta_n := (\tau \circ \epsilon_n \circ \eta_{X/S} \circ F_n)(\xi) \in H^m(X, \mathbb{Z}/n \otimes \Omega^*_{X/S});$$

note that $H^m(X, \mathbb{Z}/n \otimes \Omega^*_{X/S}) = H^m_{\mathrm{dR}}(X/S) \otimes_R R/n$, because the Hodge to de Rham spectral sequence degenerates at E_1 and hence $H^*_{\mathrm{dR}}(X/S)$ is a free R-module. Recall that $H^{r-m}(X, \Omega^r_{X/S}) \subset H^{2r-m}_{\mathrm{dR}}(X/S)$, as a next step we need to prove

$$\langle \zeta_n, \omega_j^\vee \rangle = B_{n,j} \mod nR, \tag{4.2.9}$$

via the pairing

$$\langle ., . \rangle : H^m_{\mathrm{dR}}(X/S) \times H^{2r-m}_{\mathrm{dR}}(X/S) \to H^{2r}_{\mathrm{dR}}(X/S) = H^r(X, \Omega^r_{X/S}) \xrightarrow{\mathrm{Tr}} R.$$

Indeed, we have

$$\langle \zeta, \omega_j^\vee \rangle = (q(\zeta), \omega_j^\vee) \quad \text{for all } \zeta \in H^m_{\mathrm{dR}}(X/S),$$

with $q : H^m_{\mathrm{dR}}(X/S) \to H^m(X, \mathcal{O}_X)$ induced by the projection $\Omega^*_{X/S} \to \mathcal{O}_X$, and $(., .)$ is the Grothendieck duality pairing (4.2.3). Now the equality $q(\zeta_n) = \pi(F_n\xi)$ modulo nR implies (4.2.9).

Since ζ_n is the image of the class $(\tau \circ \epsilon_n \circ \eta_{X/\mathbb{Z}} \circ F_n)(\xi) \in H^m(X, \mathbb{Z}/n \otimes \Omega^*_{X/\mathbb{Z}})$, it is horizontal for the Gauss-Manin connection. Therefore $D(\zeta_n) = 0$ for all derivations D of R. From the compatibility of $\langle ., . \rangle$ with the Gauss-Manin connection we get

$$D(B_{n,j}) = \langle D(\zeta_n), \omega_j^\vee \rangle + \langle \zeta_n, D(\omega_j^\vee) \rangle = \langle \zeta_n, D(\omega_j^\vee) \rangle \mod nR$$

for all derivations. This implies the theorem. □

Example 4.2.13. The logarithm of the formal group $\Phi^{n-r}(X/R)$ attached to a complete intersection $X \subset \mathbb{P}^n_R$ of codimension r has been computed by Stienstra [Sti87].

The classical example for Theorem 4.2.5 is the Legendre family $X = \{zy^2 - x(x-z)(x-\lambda z) = 0\} \subset \mathbb{P}^2_R$ of elliptic curves, with $R = \mathbb{Z}[\lambda][\frac{1}{2\lambda(1-\lambda)}]$. An invariant one-form of $\Phi^1(X/R)$ is given by

$$\omega = \sum_{\substack{n=0 \\ n \text{ even}}}^{\infty} \binom{n}{\frac{n}{2}} \sum_{k=0}^{\frac{n}{2}} \binom{\frac{n}{2}}{k}^2 \lambda^k x^n dx$$

with the coordinate x from [Sti87] ($\Phi^1(X/R)$ is one dimensional). As D'_R-module, $H^1_{\mathrm{dR}}(X/R)$ is annihilated by

$$D := \lambda(1-\lambda)(\frac{d}{d\lambda})^2 + (1-2\lambda)\frac{d}{d\lambda} - \frac{1}{4}$$

[Kat84]. Since $\binom{n}{\frac{n}{2}} \equiv \pm 1 \mod n+1$, Theorem 4.2.5 implies that

$$D(\sum_{k=0}^{\frac{n}{2}} \binom{\frac{n}{2}}{k}^2 \lambda^k) = 0 \mod n+1,$$

which the reader may prove easily by a direct computation (see [Kat84]). The sequence $(\sum_{k=0}^{\frac{n}{2}} \binom{\frac{n}{2}}{k}^2 \lambda^k)_{n+1}$ forms an element in

$$F \in \varprojlim_{\substack{n+1 \\ n \text{ even}}} \mathbb{Z}/(n+1)[\lambda] \subset (\prod_{p \neq 2} \mathbb{Z}_p)[|\lambda|]$$

(the projective system is ordered by division) that satisfies $D(F) = 0$ and $F(0) = 1$; thus it is given by the hypergeometric series $F = {}_2F_1(\frac{1}{2}, \frac{1}{2}; 1, \lambda)$. Since F suffices to reconstruct the operator D, the quotient morphism from Corollary 4.2.9 is an isomorphism in this case.

The analogous statements can also be proved for the Dwork family $\{\sum_{i=0}^{m} x_i^{m+1} - (m+1)\lambda x_0 \cdots x_m = 0\} \subset \mathbb{P}^m_R$ of Calabi-Yau varieties (see [Yu09]).

4.3. **Formal groups arising from varieties in characteristic zero.**

4.3.1. Let K be a field of characteristic zero. We define

$$\Sigma_K := \{A \subset K \mid A \text{ is a subring and finitely generated over } \mathbb{Z}\}.$$

For $A, B \in \Sigma_K$ there is $C \in \Sigma_K$ with $A \subset C$ and $B \subset C$.

Definition 4.3.2. We define a category FL_K with objects $\mathfrak{X}_A$ where $\mathfrak{X}_A$ is a commutative formal Lie group over A, and $A \in \Sigma_K$. For two objects $\mathfrak{X}_A, \mathfrak{Y}_B$ we set

$$\mathrm{Hom}_{FL_K}(\mathfrak{X}_A, \mathfrak{Y}_B) = \varinjlim_{\substack{C \in \Sigma_K \\ C \supset A, B}} \mathrm{Hom}(\mathfrak{X}_A \otimes_A C, \mathfrak{Y}_B \otimes_B C).$$

The Hom on the right hand side is taken in the category of formal groups over C.

For $\mathfrak{X}_A \in \mathrm{ob}(FL_K)$ and $C \in \Sigma_K$ such that $C \supset A$ we have a canonical isomorphism $\mathfrak{X}_A \otimes_A C \cong \mathfrak{X}_A$ in FL_K.

4.3.3. The category FL_K is additive. Moreover, for $\mathfrak{X} \in \mathrm{ob}(FL_K)$ and $n \in \mathbb{Z}\backslash\{0\}$, $n \in \mathrm{End}_{FL_K}(\mathfrak{X})$ is invertible.

We have an additive functor:

$$T : FL_K \to (\text{finite dimensional } K\text{-vector spaces})$$
$$T(\mathfrak{X}_A) := T\mathfrak{X}_A \otimes_A K.$$

The functor T is faithful because assigning the tangent space is a faithful functor for commutative formal Lie groups.

4.3.4. Our next goal is to show that FL_K is an abelian category. For this we need to translate Lemma 3.2.2 and 3.2.3 into the language of formal groups.

Lemma 4.3.5. *Let $g : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of commutative formal Lie groups over a ring R. Suppose that $T\mathfrak{X}$ and $T\mathfrak{Y}$ are free R-modules and Tg is surjective. Then* $\ker(g)$ *exists in the category of commutative formal Lie groups. Moreover, the following sequence is exact:*

$$0 \to T\ker(g) \to T\mathfrak{X} \to T\mathfrak{Y} \to 0.$$

Lemma 4.3.6. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of commutative formal Lie groups over a ring R which is flat over $\mathbb{Z}$. Suppose that $T\mathfrak{X}$, $T\mathfrak{Y}$, and* $\mathrm{coker}(Tf)$ *are free R-modules. Then $\mathfrak{Z} :=$* $\mathrm{coker}(f)$ *exists in the category of commutative formal Lie groups. Moreover, the following sequence is exact:*

$$T\mathfrak{X} \to T\mathfrak{Y} \to T\mathfrak{Z} \to 0.$$

The proof of Lemma 4.3.5 and 4.3.6 is an easy application of Theorem 2.3.8 and Proposition 2.3.10.

Proposition 4.3.7. *The category FL_K is an abelian category. The functor T is exact.*

Proof. Let $f \in \mathrm{Hom}_{FL_K}(\mathfrak{X}_A, \mathfrak{Y}_B)$, we claim that $\ker(f)$ and $\mathrm{coker}(f)$ exist in FL_K.

We may assume that $A = B$ and f is induced by $\phi \in \mathrm{Hom}(\mathfrak{X}_A, \mathfrak{Y}_A)$, ϕ being a morphism in the category of commutative formal Lie groups over A. Shrinking $\mathrm{Spec}(A)$ we may assume that $\mathrm{coker}(T\phi)$ is a free A-module of finite rank. Lemma 4.3.6 implies that $\mathrm{coker}(\phi)$ exists. For $C \in \Sigma_K$ with $C \supset A$ we have $\mathrm{coker}(\phi) \otimes_A C = \mathrm{coker}(\phi \otimes_A C)$ and therefore $\mathrm{coker}(f) = \mathrm{coker}(\phi)$. Moreover, $T\mathrm{coker}(f) = \mathrm{coker}(Tf)$.

In view of Lemma 4.3.5, $\mathrm{im}(\phi) := \ker(\mathfrak{Y}_A \to \mathrm{coker}(\phi))$ exists, hence $\ker(\mathfrak{X}_A \to \mathrm{im}(\phi)) = \ker(\phi)$ exists; again we have $T\ker(\phi) = \ker(T\phi)$. Since $\ker(\phi) \otimes_A C = \ker(\phi \otimes_A C)$ for all $C \in \Sigma_K$ with $C \supset A$, we conclude that $\ker(f)$ exists and $T\ker(f) = \ker(Tf)$.

Finally, we need to show that $u : \mathrm{coim}(f) \to \mathrm{im}(f)$ is an isomorphism. We know that Tu is an isomorphism, thus there is $A \in \Sigma_K$ such that u is represented by a morphism ψ of formal Lie groups over A with $T\psi$ being an isomorphism. If $T\psi$ is an isomorphism then ψ, hence u, is an isomorphism. $\square$

Proposition 4.3.8. *Let K be a field of characteristic zero. Let $\mathfrak{X} \in FL_K$ and $q \geq 0$. There exists a functor*

$$\Phi^q(-, \mathfrak{X}) : \textit{(projective schemes over } K)^{op} \to FL_K$$

with the following property. Let $A \in \Sigma_K$ be such that $\mathfrak{X}$ is represented by a commutative formal Lie group $\mathfrak{X}_A$ over A. For any $f : X \to \mathrm{Spec}(A)$ flat and projective with $R^i f_* \mathcal{O}_X$ locally free for all i, we have

$$\Phi^q(X \otimes_A K, \mathfrak{X}) = \Phi^q(X/\mathrm{Spec}(A), \mathfrak{X} \times_{\mathrm{Spec}(A)} X). \tag{4.3.1}$$

Proof. For a morphism $f : X \to \mathrm{Spec}(K)$ we can spread out to $f' : X' \to \mathrm{Spec}(A)$ for some $A \in \Sigma_K$ where $\mathfrak{X}$ is defined and the higher direct images $R^i f_* \mathcal{O}_{X'}$ are locally free. Then we can use (4.3.1) to define $\Phi^q(-, \mathfrak{X})$. $\square$

Appendix A. Commutative formal Lie groups

In this section we recall some basic facts in the theory of commutative formal Lie groups, that we need in the main text of the paper.

A.1. **Invariant 1-forms.**

A.1.1. Let S be a scheme and $\mathfrak{X}$ be a commutative formal Lie group over S (see 1.1.7).

Definition A.1.2. A 1-form $\omega \in \Omega_{\mathfrak{X}/S}$ is called *invariant* if

$$m^*(\omega) = \mathrm{pr}_1^*(\omega) + \mathrm{pr}_2^*(\omega),$$

where $m : \mathfrak{X} \times_S \mathfrak{X} \to \mathfrak{X}$ is the multiplication and pr_i are the projections.

For example, dx is an invariant 1-form for $\hat{\mathbb{G}}_a$, and $dx/(1-x)$ is an invariant 1-form for $\hat{\mathbb{G}}_m$ in the coordinates from 1.1.1. Obviously, the invariant 1-forms form a $\mathcal{O}_S$-submodule of $\Omega_{\mathfrak{X}/S}$.

In 1.1.7 we introduced the tangent space of $\mathfrak{X}$ by

$$T\mathfrak{X}(U) := \ker(\mathfrak{X}(\mathrm{Spec}(\mathcal{O}_U[\epsilon]/\epsilon^2)) \to \mathfrak{X}(U)),$$

where $U \to \operatorname{Spec}(\mathcal{O}_U[\epsilon]/\epsilon^2)$ is the morphism over U given by $\epsilon \mapsto 0$. Given $\xi \in T\mathfrak{X}(U)$, we can define a derivation $D_\xi : \Gamma(U, \mathcal{O}_\mathfrak{X}) \to \Gamma(U, \mathcal{O}_S)$ by

$$D_\xi(f)\epsilon = \xi^*(f) - f(0).$$

Here and in the following we consider $\mathcal{O}_S$ as an $\mathcal{O}_\mathfrak{X}$-module via the zero section $f \mapsto f(0)$. This induces an isomorphism of $\mathcal{O}_S$-modules

$$T\mathfrak{X} \to \mathit{Der}_{\mathcal{O}_S}(\mathcal{O}_\mathfrak{X}, \mathcal{O}_S) = \mathit{Hom}_{\mathcal{O}_\mathfrak{X}}(\Omega_{\mathfrak{X}/S}, \mathcal{O}_S). \tag{A.1.1}$$

Proposition A.1.3. *There is a unique morphism of $\mathcal{O}_S$-modules*

$$\eta : T\mathfrak{X}^\vee \to \Omega_{\mathfrak{X}/S},$$

satisfying the following properties:

(1) *The morphism η induces an isomorphism between $T\mathfrak{X}^\vee$ and the invariant 1-forms.*
(2) *The diagram*

$$\begin{tikzcd} T\mathfrak{X}^\vee \times T\mathfrak{X} \arrow[r] \arrow[d, "\eta\times id"] & \mathcal{O}_S \arrow[d, "="] \\ \Omega_{\mathfrak{X}/S} \times T\mathfrak{X} \arrow[r, "(A.1.1)"] & \mathcal{O}_S \end{tikzcd}$$

is commutative.

Proof. [Zin84, 1.19] □

If $\mathfrak{X} = \operatorname{Spf}(\mathcal{O}_S[|x_1, \dots, x_d|])$ as formal scheme over S then

$$\Omega_{\mathfrak{X}/S} \to \mathcal{O}_S^d, \quad \sum_{i=1}^{d} f_i dx_i \mapsto (f_1(0), \dots, f_d(0)),$$

induces an isomorphism between the invariant 1-forms and $\mathcal{O}_S^d$.

Lemma A.1.4. *Let S be a scheme that is flat over $\operatorname{Spec}(\mathbb{Z})$. Let $\mathfrak{X}$ be a commutative formal Lie group over S. Then the invariant 1-forms are closed.*

Proof. We may suppose $S = \operatorname{Spec}(R)$. Denoting by $S \otimes_\mathbb{Z} \mathbb{Q}$ the base change to $\operatorname{Spec}(\mathbb{Q})$, we have $\Gamma(S, \Omega^2_{\mathfrak{X}/R}) \subset \Gamma(S \otimes_\mathbb{Z} \mathbb{Q}, \Omega^2_{\mathfrak{X}\otimes_\mathbb{Z}\mathbb{Q}/R\otimes_\mathbb{Z}\mathbb{Q}})$. Since $\mathfrak{X} \otimes_\mathbb{Z} \mathbb{Q}$ is isomorphic to the formal completion of a vector bundle (Section A.2) we are reduced to the obvious case $\hat{\mathbb{G}}_a$. □

A.2. Logarithms and formal groups in characteristic zero.

A.2.1. Let S be a scheme. Recall from Section 1.1.7 that we have a functor
(A.2.1)
T : (Comm. formal Lie groups over S) $\to$ (loc. free sheaves on S of finite rank),

that assigns to a group its tangent space (or Lie algebra).

Proposition A.2.2. *Suppose that S is a scheme over $\operatorname{Spec}(\mathbb{Q})$. Then T (A.2.1) is an equivalence of categories.*

Proof. [Zin84, 4.7] □

For every locally free sheaf $\mathcal{E}$ of finite rank on S we have the associated vector bundle

$$V(\mathcal{E}) := \mathrm{Spec}(\mathrm{Sym}^*\mathcal{E}^\vee).$$

The formal completion $\hat{V}(\mathcal{E})$ yields a commutative formal Lie group with tangent space $\mathcal{E}$. If S is a scheme over $\mathbb{Q}$, then for every commutative formal Lie group $\mathfrak{X}$ there exists a unique isomorphism of formal groups

$$\mathfrak{X} \to \hat{V}(T\mathfrak{X}),$$

such that the induced map on the tangent spaces is the identity; the functor $\hat{V}$ is inverse to T. For every morphism $\phi : T\mathfrak{X} \to \mathcal{E}$ we will denote by

$$\log_\phi : \mathfrak{X} \to \hat{V}(\mathcal{E})$$

the unique morphism over ϕ.

In general we identify $\hat{V}(\mathcal{O}_S^d)$ with $\hat{\mathbb{G}}_a^d$. A basic example is $\mathfrak{X} = \hat{\mathbb{G}}_m$, $\mathcal{E} = \mathcal{O}_S$, and ϕ coming from the coordinate of $\hat{\mathbb{G}}_m$ in 1.1.1. Then $\log_\phi : \hat{\mathbb{G}}_m \to \hat{\mathbb{G}}_a$ is given by $\log_\phi^*(x) = \sum_{k=1}^\infty \frac{x^k}{k}$.

A.2.3. Again, suppose that S is a scheme over $\mathrm{Spec}(\mathbb{Q})$. The canonical morphism $\mathfrak{X} \to \hat{V}(T\mathfrak{X})$ can be constructed as follows. By Proposition A.1.3 we have a canonical morphism $\eta : T\mathfrak{X}^\vee \to \Omega_{\mathfrak{X}/S}$ that induces an isomorphism with the invariant forms. Since invariant forms are closed (Lemma A.1.4) we obtain a morphism of $\mathcal{O}_S$-modules

$$d^{-1} \circ \eta : T\mathfrak{X}^\vee \to \ker(0^* : \mathcal{O}_\mathfrak{X} \to \mathcal{O}_S).$$

This morphism induces $\mathfrak{X} \to \hat{V}(T\mathfrak{X})$.

A.2.4. Let $S = \mathrm{Spec}(R)$ be flat over $\mathrm{Spec}(\mathbb{Z})$. Let $\mathfrak{X}$ be a commutative formal Lie group. Suppose that $\mathfrak{X} = \mathrm{Spf}(R[|x_1, \ldots, x_d|])$ as formal schemes and that the zero is defined by $x_i \mapsto 0$. We denote by $\mathfrak{X}_\mathbb{Q}$ the base change to $S_\mathbb{Q} := S \times_{\mathrm{Spec}(\mathbb{Z})} \mathrm{Spec}(\mathbb{Q})$.

Since $T\mathfrak{X}$ comes equipped with a trivialization, we have a canonical map

$$\mathfrak{X}(R[|x|]) \to \mathfrak{X}_\mathbb{Q}((R\otimes\mathbb{Q})[|x|]) \xrightarrow{\log} \hat{\mathbb{G}}_a^d((R\otimes\mathbb{Q})[|x|]) = \bigoplus_{i=1}^d x \cdot (R\otimes\mathbb{Q})[|x|]. \tag{A.2.2}$$

References

[AM77] M. Artin and B. Mazur. Formal groups arising from algebraic varieties. *Ann. Sci. École Norm. Sup. (4)*, 10(1):87–131, 1977.

[BBE07] Pierre Berthelot, Spencer Bloch, and Hélène Esnault. On Witt vector cohomology for singular varieties. *Compos. Math.*, 143(2):363–392, 2007.

[Hes] Lars Hesselholt. The big de Rham-Witt complex. arXiv:1006.3125v1.

[HM01] Lars Hesselholt and Ib Madsen. On the K-theory of nilpotent endomorphisms. In *Homotopy methods in algebraic topology (Boulder, CO, 1999)*, volume 271 of *Contemp. Math.*, pages 127–140. Amer. Math. Soc., Providence, RI, 2001.

[Hon70] Taira Honda. On the theory of commutative formal groups. *J. Math. Soc. Japan*, 22:213–246, 1970.

[Kat84] Nicholas M. Katz. Expansion-coefficients as approximate solution of differential equations. *Astérisque*, (119-120):6, 183–189, 1984. p-adic cohomology.

[KO68] Nicholas M. Katz and Tadao Oda. On the differentiation of de Rham cohomology classes with respect to parameters. *J. Math. Kyoto Univ.*, 8:199–213, 1968.

[Sti87] Jan Stienstra. Formal group laws arising from algebraic varieties. *Amer. J. Math.*, 109(5):907–925, 1987.

[Sti91] Jan Stienstra. The generalized de Rham-Witt complex and congruence differential equations. In *Arithmetic algebraic geometry (Texel, 1989)*, volume 89 of *Progr. Math.*, pages 337–358. Birkhäuser Boston, Boston, MA, 1991.

[Yu09] Jeng-Daw Yu. Variation of the unit root along the Dwork family of Calabi-Yau varieties. *Math. Ann.*, 343(1):53–78, 2009.

[Zin84] Thomas Zink. *Cartiertheorie kommutativer formaler Gruppen*, volume 68 of *Teubner-Texte zur Mathematik [Teubner Texts in Mathematics]*. BSB B. G. Teubner Verlagsgesellschaft, Leipzig, 1984. With English, French and Russian summaries.

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